Stat 543 3-25-05

<u>Recall</u> MLR $\theta_1 < \theta_2$

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} \uparrow \text{ in } T(x)$$

<u>Example</u> single parameter exponential families in natural parametrization ... for a single observation

$$f(x|\eta) = h(x) \exp\left(\eta T(x) - A(\eta)\right)$$

$X_1, X_2, \ldots, X_n$ iid $f(x|\eta)$

$$f(x|\eta) = \prod_{i=1}^{n} h(x_i) \exp\left(\eta \sum T(x_i) - nA(\eta)\right)$$

The likelihood ratio for $\eta_1 < \eta_2$ is

$$\frac{f(x|\eta_2)}{f(x|\eta_1)} = \exp\left[(\eta_2 - \eta_1)\sum T(x_i) - nA(\eta_2) + nA(\eta_1)\right]$$

and this is "clearly" monotone nondecreasing in $\sum T(x_i)$ so we have MLR in $\sum T(X_i)$

<u>Example</u> single parameter exponential family with marginal pdf or pmf

$$f(x|\theta) = h(x)\exp\left[\eta(\theta)T(x) - A(\eta(\theta))\right]$$

if $\eta(\theta)$ is increasing in θ we have again that the family has MLR in $\sum T(X_i)$

— Note BTW that if

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} \text{ is nonincreasing in } T(x)$$

one has MLR in $-T(X)$

— There are a number of convenient mathematical properties that follow from MLR – for one, if one assumes that $\Theta \sim g(\theta)$ and there is MLR in X (or $-X$), $E[\Theta | X = x]$ (the posterior mean) is monotone in x –

— What's of most interest here is a theorem about UMP testing

Theorem Suppose that the family of dsns specified by $f(x|\theta)$ has MLR in a statistic $T(X)$. Then for any $\alpha \in (0,1]$ $\exists$ a UMP size α test of

$$\text{A} \quad \begin{array}{l} H_0: \theta \leq \theta_0 \\ H_a: \theta > \theta_0 \end{array} \qquad \left(\text{or} \quad \text{B} \quad \begin{array}{l} H_0: \theta \geq \theta_0 \\ H_a: \theta < \theta_0 \end{array} \right)$$

of the form

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > k \quad (<) \\ \nu & \phantom{\text{if } T(x)} = \\ 0 & \text{if } T(x) < k \quad (>) \end{cases}$$

with $k \in (-\infty, \infty)$ and $\nu \in [0,1]$. Further,

$$\pi_\phi(\theta_0) = \alpha$$

The basic idea is the MLR allows $T(x)$ to function in place of every likelihood ratio

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} \quad \text{for } \theta_2 > \theta_1$$

Example $X_1, X_2, \ldots, X_{10}$ iid Poisson λ

Consider testing $H_0: \lambda \geq 1$ vs $H_a: \lambda < 1$ — note that since for a single X_i

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{x!}\exp\left(x\log\lambda - \lambda\right)$$

and $\log\lambda$ is increasing in λ $\therefore$ i.e. we're in an exponential family and have MLR in $\sum_{i=1}^{10} X_i$

and the theorem says that UMP tests of our hypotheses (which are of form B) can be gotten as

$$\phi(x) = \begin{cases} 1 & \text{if } \sum x_i < k \\ \nu & \qquad = \\ 0 & \text{if } \sum x_i > k \end{cases}$$

and the size of such a test $\pi_\phi(1)$ — for sake of example, let's invent the UMP size $\alpha = .005$ test — using the fact that

$\sum X_i \sim \text{Poisson}(10\lambda)$ if $\lambda = 1$

$$P_{\lambda=1}[\sum X_i \le 0] \approx .000$$

$$P_{\lambda=1}[\sum X_i \le 1] \approx .000$$

$$P_{\lambda=1}\left[\sum X_i \le 2\right] \approx .003$$

$$P_{\lambda=1}\left[\sum X_i \le 3\right] \approx .010$$

$$P_{\lambda=1}\left[\sum X_i \le 4\right] \approx .029$$

recalling that I want $\alpha = .005$ (and that α is the $\lambda=1$ power of the test) I need to reject if $\sum X_i < 3$, accept if $\sum X_i > 3$ and randomize if $\sum X_i = 3$ —

$$P_{\lambda=1}\left[\sum X_i = 3\right] \approx .010 - .003 = .007$$

i.e. I'll use

$$\phi(x) = \begin{cases} 1 & \text{if } \sum x_i < 3 \\ \frac{2}{7} & \qquad\;\; = \\ 0 & \text{if } \sum x_i > 3 \end{cases}$$

and the theorem guarantees that

$$\alpha = \sup_{\lambda \geq 1} \Pi_\phi(\lambda) = \Pi_\phi(1) = E_{\lambda=1}\phi(X)$$

$$= P_{\lambda=1}\left[\sum X_i < 3\right]$$

$$+ \frac{2}{7} P_{\lambda=1}\left[\sum X_i = 3\right]$$

$$= .003 + \frac{2}{7}(.007)$$

$$= .005$$

and that this test is UMP of size <= .005

Bottom Line on UMP testing

1-parameter

① Primarily available in MLR families
where one is testing <u>one-sided</u> alternatives

one can not even get UMP tests in MLR families for stat 101 type 2-sided alternatives ☹

$$X \sim N(\mu, 1) \qquad H_0: \mu = 0 \quad \text{vs} \quad H_a: \mu \neq 0$$

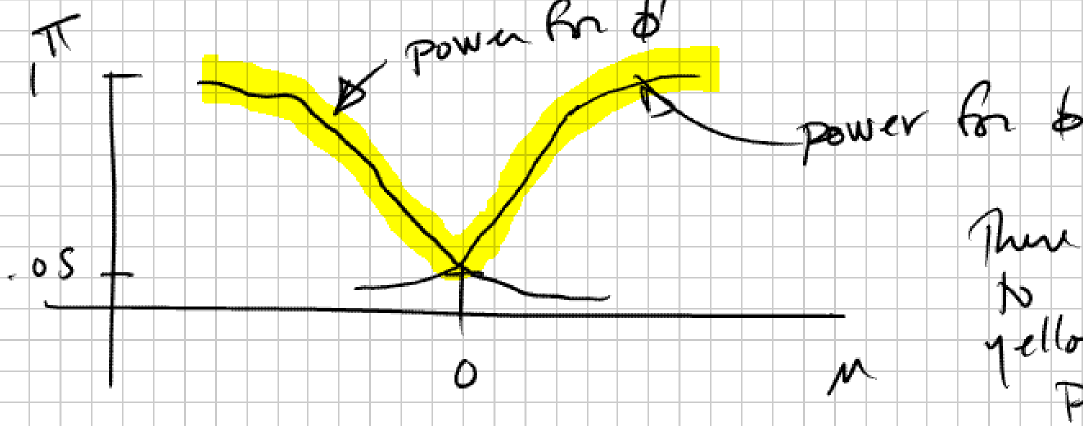
There is no UMP test here

$$\phi(x) = I[x > 1.645]$$

has largest power at any $\mu > 0$ for a test that has
$\pi_\phi(0) = .05$

$\phi'(x) = I[x < -1.645]$ has largest power at any $\mu < 0$ for a test with $\pi_{\phi'}(0) = .05$

(and a uniqueness part of N-P that I didn't state would say that these are the only tests that achieve these maximal power values)



There is no way to achieve the yellow profile on power