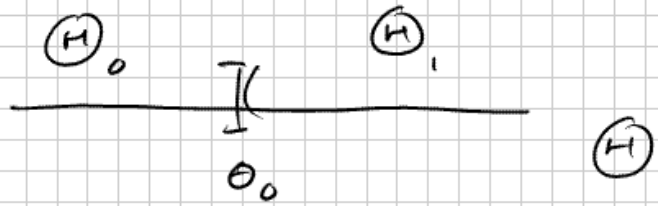
Stat 543 3-23-05

Non-Bayesian Optimality Thy for testing? (beyond simple vs simple) – here about all that can be done in satisfying fashion is for some one-sided problem where the parameter space is one-dimensional



$H_0: \theta \le \theta_0$ vs

$H_1: \theta > \theta_0$

once I leave simple vs simple I need to extend the notion of "size" for a test and rethink what "optimality" would mean

<u>Def</u> The <u>size</u> of a test $\phi(x)$

$$\text{"max"}_{\theta \in \Theta_0} E_\theta \phi(X) = \text{"max"}_{\theta \in \Theta_0} \pi_\phi(\theta)$$

what I would like is to be able to look around in set of test ϕ with size $\leq \alpha$ and pick out one that has maximum power for every $\theta \in \Theta_1$

<u>Def</u> A test $\phi(x)$ is UMP of size α (uniformly most powerful) provided

1) $\pi_\phi(\theta) \leq \alpha \quad \forall\, \theta \in \Theta_0$

and 2) for any other test ϕ' of size $\leq \alpha$

$$\pi_\phi(\theta) \geq \pi_{\phi'}(\theta) \quad \forall \ \theta \in \Theta_1$$

What can be done in this direction can be best introduced in terms of a motivating example

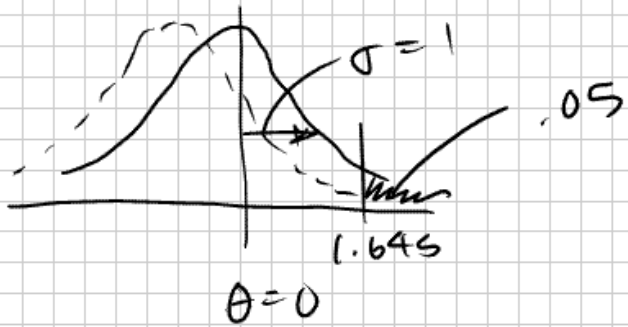
<u>Example</u> $X \sim N(\theta, 1)$ testing $H_0: \theta \leq 0$ vs $H_1: \theta > 0$ - suppose I'm interested in tests of size $\alpha \leq .05$ - a plausible test here is

$$\phi(x) = \begin{cases} 1 & \text{if } x > 1.645 \\ 0 & \quad < \end{cases}$$

as it turns out, this test is UMP size $\alpha = .05$ for testing $H_0: \theta \leq 0$ vs $H_1: \theta > 0$ ....

How to show this?

<u>Step 1</u> Note that ϕ is of size .05



$$\pi_\phi(\theta) = 1 - \Phi(1.645 - \theta)$$

for $\theta \le 0 \rightarrow \le \; 1 - \Phi(1.645 - 0)$
$= .05$

i.e. $\sup_{\theta \le 0} \pi_\phi(\theta) = \pi_\phi(0) = .05$

<u>Step 2</u> Note that for $\theta_1 > 0$, ϕ is a MP level $\alpha = .05$ test of $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$

Why?

$$\frac{f(x|\theta_1)}{f(x|0)} = \frac{\frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x^2 - 2\theta_1 x + \theta_1^2)}{\frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}x^2}$$

$= R(x)$

$$= \exp\left(\theta_1 x - \frac{1}{2}\theta_1^2\right) \nearrow \text{ in } x$$

since $\theta_1 > 0$

i.e. ϕ is of N-P form for $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$

<u>Step 3</u> Note that Step 2 implies that ϕ is UMP size $\alpha = .05$ for $H_0: \theta = 0$ vs $H_1: \theta > 0$

Why? NP says that any other test ϕ' with

$$\pi_{\phi'}(0) \le \pi_\phi(0) = .05$$

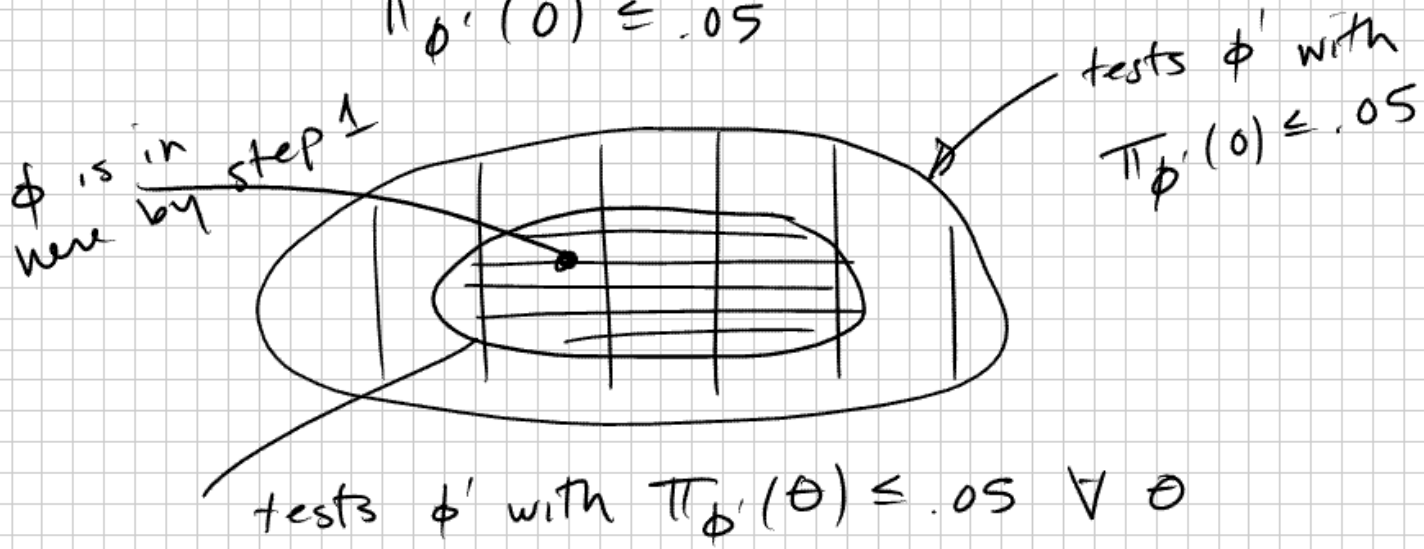
has $\pi_{\phi'}(\theta_1) \le \pi_\phi(\theta_1)$ for any $\theta_1 > 0$

<u>Step 4</u> recognize that the set of tests ϕ' with

$$\pi_{\phi'}(\theta) \le .05 \quad \forall\ \theta \le 0$$

is a subcollection of the set of tests ϕ' with

$$\pi_{\phi'}(0) \le .05$$



<u>Step 5</u> Since from step 3, if ϕ' is in

it must be the case that

$$\pi_{\phi'}(\theta) \le \pi_{\phi}(\theta) \qquad \forall \ \theta > 0$$

it is also true that if ϕ' is in

$$\pi_{\phi'}(\theta) \le \pi_{\phi}(\theta) \qquad \forall \ \theta > 0$$

(the toughest guy in town is the toughest guy in his neighborhood)

This is exactly that ϕ is UMP level $\alpha = .05$ for $H_0: \theta \le 0$ vs $H_1: \theta > 0$

This is an important example ... what makes it work is that for any $\theta_0 < \theta_1$

$$R(x) = \frac{f(x|\theta_1)}{f(x|\theta_0)}$$

is increasing in x so that rejecting for large x substitutes for rejecting for large $R(x)$

lets me apply N-P

This can be generalized exactly this far:

Consider $\Theta \subset \mathbb{R}^1$ and hypotheses

A $H_0: \theta \le \theta_0$, $H_1: \theta > \theta_0$ or B $H_0: \theta \ge \theta_0$, $H_1: \theta < \theta_0$

where X has pdf or pmf $f(x|\theta)$

<u>Def</u> The family of distributions specified by $f(x|\theta)$ has Montone Likelihood Ratio in a statistic $T(X)$ provided for $\theta_1 < \theta_2$ belonging to Θ

$$\frac{f(x|\theta_2)}{f(x|\theta_1)}$$

is nondecreasing in $T(x)$ $\left(\text{on } \{x \mid f(x|\theta_1) + f(x|\theta_2) > 0\}\right)$

<u>Example</u> $N(\mu, 1)$ for $\mu_1 < \mu_2$

$$\frac{f(x|\mu_2)}{f(x|\mu_1)} = \exp -\tfrac{1}{2}\left[(x-\mu_2)^2 - (x-\mu_1)^2\right]$$

$$= \exp\left[-\tfrac{1}{2}(\mu_2^2-\mu_1^2) + x(\mu_2-\mu_1)\right]$$

in x

so we have MLR in X

<u>Example</u> Binomial (n,p) for $p_1 < p_2$

$$\frac{f(x|p_2)}{f(x|p_1)} = \left(\frac{p_2}{p_1}\right)^x \left(\frac{1-p_2}{1-p_1}\right)^{n-x}$$

$$= \left[\frac{p_2/(1-p_2)}{p_1/(1-p_1)}\right]^x \left(\frac{1-p_2}{1-p_1}\right)^n$$

This is increasing x i.e. The family of Binomial dsns has MLR in p