Stat 543 3-21-05

Recall : $\phi : \mathcal{X} \longrightarrow [0,1]$ a "randomized test"

<u>Example</u> Simple Discrete one

only a few α's available for non-randomized tests ... randomization gives others

$$\phi(x) = \begin{cases} 1 & \text{if } R(x) > 2 \\ .25 & \text{if } R(x) = 2 \\ 0 & \text{if } R(x) < 2 \end{cases}$$

size $\alpha = .1$

this is of N-P form and is MP of size $\alpha = .1$

In fact there is a general story motivated by this simple discrete example

Thm (N-P Lemma) (Part II) (Existence)
If α $(0,1]$ $\exists$ a test of the form

$$\phi(x) = \begin{cases} 1 & \text{if } R(x) > k \\ \gamma & \quad\quad = \\ 0 & \quad\quad < \end{cases}$$

for some $0 \le k < \infty$ and $0 \le \gamma \le 1$ that is most powerful of size α for testing $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$

(The construction I used in the discrete example can be made in general)

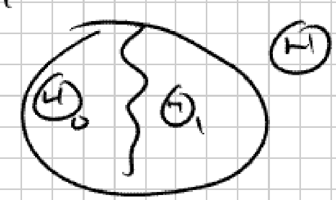
details of proof are not hard (imitate ape the example)
but are tedious ... not worth class time

So what kind of theory is available for problems more complicated than simple vs simple?

Bayes with 0-1 loss I can give a complete answer to "what is optimal?"

Theoretically complete but possibly hard to implement

For a prior $\theta \sim g(\theta)$ a Bayes test of
$H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$



is

$$\phi(x) = \begin{cases} 1 & \text{if } P[\theta \in \Theta_1 \mid X = x] > \frac{1}{2} \\ 0 & < \end{cases}$$

$$= \begin{cases} 1 & \text{if } \dfrac{P[\theta \in \Theta_1 \mid X = x]}{P[\theta \in \Theta_0 \mid X = x]} > 1 \\ 0 & < \end{cases}$$

Further, (e.g. in $\text{cont}^{\underline{s}}$ cases) This is

$$\frac{\int_{\Theta_1} L_x(\theta) g(\theta)\, d\theta}{\int_{\Theta_0} L_x(\theta) g(\theta)\, d\theta}$$

(note that $g(\cdot)$ averages $R(x)$ of num + denom of $L_x(\theta_1)$ replace and $L_x(\theta_0)$)

Example $X \sim N(\theta, 1)$ $\quad \underbrace{\Theta \sim N(0, \gamma^2)}_{g(\theta)}$

$$\theta \mid X = x \text{ is } N\left(\frac{\gamma^2}{\gamma^2+1}x, \frac{\gamma^2}{\gamma^2+1}\right)$$

Suppose I want to test $H_0: |\theta| \le .3$ vs $H_1: |\theta| > .3$ $\left(\Theta_0 = [-.3, .3] \quad \Theta_1 = (-\infty, -.3) \cup (.3, \infty) \right)$

What is a Bayes test? Reject if posterior probability of Θ_0 is small (if posterior probability of Θ_1 is large)

$$P\left[\theta \in \Theta_0 \mid X = x\right] = \Phi\left(\frac{.3 - \frac{\gamma^2}{\gamma^2+1}x}{\sqrt{\frac{\gamma^2}{\gamma^2+1}}}\right) - \Phi\left(\frac{-.3 - \frac{\gamma^2}{\gamma^2+1}x}{\sqrt{\frac{\gamma^2}{\gamma^2+1}}}\right)$$

and a Bayes test (with 0-1 loss) just amounts to checking to see if this is less than .5 — if it is, I decide in favor of Θ_1.

__Example__ Discrete example from Exam I –

Find a Bayes test of $H_0: \theta = 1 \text{ or } 3$ vs $H_1: \theta = 2 \text{ or } 4$ with a prior with pmf

θ	1	2	3	4
$g(\theta)$	.4	.2	.2	.2

$f(x|\theta)$ given by

θ \ x	1	2	3	4	5	6
1	.05	.1	.025	.3	.375	.15
2	.2	.4	.1	.1	.15	.05
3	.1	.2	.05	.2	.35	.1
4	.05	.1	0	.4	.25	.2

$f(x|\theta)\,g(\theta)$ is given by $\qquad H_0: \theta = 1 \text{ or } 3 \qquad H_a: \theta = 2 \text{ or } 4$

θ \ x	1	2	3	4	5	6
1	.02	.04	.01	.12	.15	.06
2	.04	.08	.02	.02	.03	.01
3	.02	.04	.01	.04	.07	.02
4	.01	.02	0	.08	.05	.04

a Bayes test is

$$\phi(x) = \begin{cases} 1 & \text{if } x = 1, 2 \\ 0 & \text{if } x = 4, 5, 6 \end{cases}$$