Stat 543 3-11-05

<u>Theorem</u> (Sufficiency Part) (N-P Lemma) If $\phi(x) = \begin{cases} 1 & R(x) > k \\ 0 & < \end{cases}$

Then $\phi(x)$ is most powerful of its size

<u>Pf</u>: $\alpha = \pi_\phi(\theta_0) = P_{\theta_0}[\phi(X) = 1]$

$= E_{\theta_0}\phi(X)$

Suppose that ϕ' is a competing test with

$\pi_{\phi'}(\theta_0) \le \alpha$

and consider

$$E_{\theta_1}\left[\phi(X) - \phi'(X)\right] - k\, E_{\theta_0}\left[\phi(X) - \phi'(X)\right]$$

cont's case

$$\stackrel{\equiv}{} \int \left(\phi(x) - \phi'(x)\right) f(x|\theta_1) - k\left(\phi(x) - \phi'(x)\right) f(x|\theta_0)\, dx$$

$$= \int \left[\phi(x) - \phi'(x)\right]\left(R(x) - k\right) f(x|\theta_0)\, dx$$

Notice that when $R(x) - k > 0$ $\left(R(x) > k\right)$ $\phi(x) = 1$ and $\phi(x) - \phi'(x) \geq 0$ — When $R(x) - k < 0$ $\left(R(x) < k\right)$ and $\phi(x) - \phi'(x) \leq 0$ and overall

$$\left(\phi(x) - \phi'(x)\right)\left(R(x) - k\right) \geq 0$$

So

$$E_{\theta_1}\left[\phi(X) - \phi'(X)\right] - k\,E_{\theta_0}\left[\phi(X) - \phi'(X)\right]$$

$$= \int (\text{something non-negative})\, f(x|\theta_0)\, dx \geq 0$$

and thus

$$E_{\theta_1}\left[\phi(X) - \phi'(X)\right] \geq k\,E_{\theta_0}\left[\phi(X) - \phi'(X)\right]$$

$$\pi_\phi(\theta_1) - \pi_{\phi'}(\theta_1) \geq k\left[\underbrace{\pi_\phi(\theta_0) - \pi_{\phi'}(\theta_0)}_{\alpha\,-\,\text{size of }\phi' \,\geq\, 0}\right]$$

So $\pi_\phi(\theta_1) - \pi_{\phi'}(\theta_1) \geq 0$ ∎

No test of size no more than α has better power than such a ϕ

Example $\quad N(\mu_0, 1) \quad vs \quad N(\mu_1, 1) \quad \mu_0 < \mu_1$

$R(x) \nearrow$ in x, so rejecting for large $R(x)$ is equivalent to rejecting for large x — consider

$$\phi_c = I[X > c]$$

$$\alpha_c = P_{\mu_0}[X > c] = 1 - \Phi(c - \mu_0)$$

$$\pi_{\phi_c}(\mu_1) = 1 - \Phi(c - \mu_1)$$

for a given value of this you can't beat that

Example Simple Discrete Problem

x	0	1	2	3	4	5
$f(x \mid \theta_1)$	.2	.1	.4	.2	.05	.05
$f(x \mid \theta_0)$	.2	.1	.2	.05	.3	.15
$R(x)$	1	1	2	4	$\frac{1}{6}$	$\frac{1}{3}$

test	x's in rejection region	size/α	"power"/ $\pi_\phi(\theta_1)$
1	3	.05	.2
2	3, 2	.25	.6
3	3, 2, 1	.35	.7
4	3, 2, 0	.45	.8
5	3, 2, 0, 1	.55	.9
6	3, 2, 0, 1, 5	.70	.95

When $R(X)$ has a cont$^{\underline{s}}$ dsn under Θ_0 it's possible to adjust k and get a MP test of any desired size, α — when (as in our discrete example) $R(x)$ has a discrete dsn it seems like only a discrete set of α's are available ... unless (for sake of mathematical/theoretical completeness) one extends the notion of a test for $\phi: \mathcal{X} \longrightarrow \{0,1\}$ to the notion of a "randomized test"

<u>Def</u> A function $\phi: \mathcal{X} \longrightarrow [0,1]$ is called a randomized test (a randomized decision function)

The interpretation is that if I observe $X = x$, I choose randomly between $a=0$ and $a=1$ with probability $\phi(x)$ assigned to $a=1$

Example Simple Discrete Example

$\phi_1(x) = I[x=3]$ has size $\alpha = .05$

$\phi_2(x) = I[x=3 \text{ or } 2]$ has size $\alpha = .25$

$$\phi(x) = \begin{cases} 1 & x=3 \\ .25 & x=2 \\ 0 & x=0,1,4 \text{ or } 5 \end{cases}$$

$x=3$ have $R(x)=4$

$x=0,1,4$ or 5 have $R(x)<2$

$$P_{\theta_0}[a=1 \text{ is selected}] = P_{\theta_0}[X=3] + (.25)P_{\theta_0}[X=2]$$

$$= .05 + (.25)(.2)$$

$$= .1$$

$$P_{\theta_0}[a=1 \text{ is selected}] = E_{\theta_0}\phi(X) = \pi_\phi(\theta_0) = \alpha$$

And this test is of the form

$$\phi(x) = \begin{cases} 1 & \text{if} \quad R(x) > 2 \\ (.25 & \text{if} \quad R(x) = 2\ ) \\ 0 & \text{if} \quad R(x) < 0 \end{cases}$$

and is a Bayes test i.e. it is of N-P form — as a matter of fact the proof of N-P lemma goes through exactly as before even allowing for these randomized tests! — so $\phi(x)$ above is a MP test of size $\alpha = .1$

$$\pi_\phi(\theta_1) = .2 + .25(.4) = .3$$