Stat 543 2-4-05

"the" method of moments

last time:

$$\gamma(\theta) = h(\mu_1(\theta), \ldots, \mu_r(\theta))$$

$$\delta_n(X) = h(\hat{\mu}_{1n}, \ldots, \hat{\mu}_{rn})$$

another view: where I can write

$$\mu_1(\theta) = h_1^*(\theta)$$

$$\mu_2(\theta) = h_2^*(\theta)$$

$$\vdots$$

$$\mu_r(\theta) = h_r^*(\theta)$$

I could define a method of moments estimator of θ to be a solution to

$$\left(h_1^*(\theta), h_2^*(\theta), \ldots, h_r^*(\theta)\right) = \left(\hat{\mu}_{1n}, \hat{\mu}_{2n}, \ldots, \hat{\mu}_{rn}\right)$$

(this version is what we used in the truncated Poisson example)

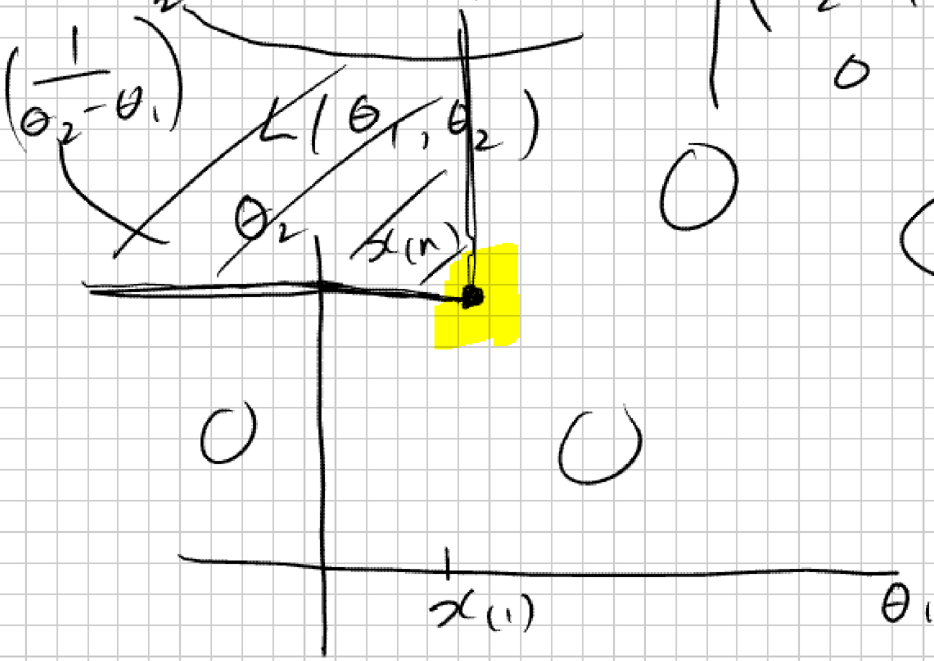
Def If $\hat{\theta}$ is a value of θ maximizing $L(\theta)$ then we will call $\hat{\theta}$ a maximum likelihood estimate of θ

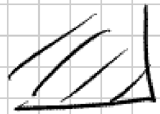
Example $X_1, \ldots, X_n$ iid $U(\theta_1, \theta_2)$

$X = (X_1, \ldots, X_n)$ with jt pdf

$$f(x \mid \theta_1, \theta_2) = \begin{cases} \left(\dfrac{1}{\theta_2 - \theta_1}\right)^n & \text{if each } x_i \in [\theta_1, \theta_2] \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_1 \le x_{(1)} \le x_{(n)} \le \theta_2$$



on  $L(\theta_1, \theta_2)$ increases in θ_1 up to $x_{(1)}$ and increases in θ_2 down to $x_{(n)}$ — so the MLE of (θ_1, θ_2)

$(x_{(1)}, x_{(n)})$

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid Poisson(λ)

$$L(\lambda) = f(x \mid \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \quad \text{on } \{0, 1, 2, \ldots\}^n$$

So $$\log L(\lambda) = -n\lambda + \left(\sum x_i\right)\log\lambda - \sum \log x_i!$$

If $\sum x_i = 0$ this is maximized by $\lambda = 0$

i.e. a maximum likelihood estimate is 0

If $\sum x_i \geq 1$

$$\frac{d}{d\lambda} \log L(\lambda) = -n + \frac{\sum x_i}{\lambda} = 0$$

implies $\lambda = \frac{\sum x_i}{n}$

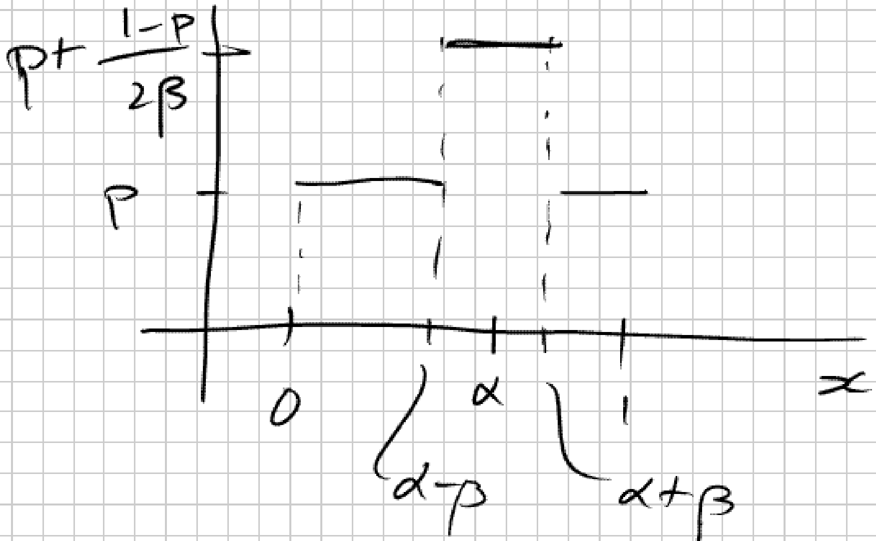
i.e. $\overline{X}$ is the MLE of the Poisson mean λ — (I really should check that $\overline{X}$ is a maximizer and not just a place where $L(\lambda) = 0$) — note that

$$\frac{d}{d\lambda} \log L(\lambda) \begin{cases} < 0 & \text{if } \lambda > \overline{x} \\ > 0 & \text{if } \lambda < \overline{x} \end{cases}$$

For a given model and observation there need not be a maximizer of $f(x|\theta)$ . . . . and if there is one there is nothing that guarantees uniqueness — in the case that $f(x|\theta)$ is a density the likelihood can be unbounded — even when it is bounded the sup of $L(\theta)$ may not be attained

Example $X_1, X_2, \ldots, X_n$ iid with marginal pdf

$$f(x \mid p, \alpha, \beta) = p\,I\left[0 \le x \le 1\right] + \frac{1-p}{2\beta} I\left[\alpha - \beta \le x \le \alpha + \beta\right]$$



with probability 1 (for any p, α, β)

$$0 < x_{(1)} \leq x_{(n)} < 1$$

and for example with $\alpha = x_{(1)}$ and $\beta < \min(x_{(1)}, x_{(2)} - x_{(1)})$ and $p > 0$

The likelihood (jt density of n observations)

is

$$L(p,\alpha,\beta) = p^{n-1}\left(p + \frac{1-p}{2\beta}\right) \nearrow \infty \quad \text{as } \beta \to 0$$

$\alpha \uparrow x_{(1)}$

So there really is no MLE here ... ☹

a "fix" to this might be to replace densities with pmf's admitting that observations are really only good to some number of decimal places ...

unfortunately, even if I "fix" the problem of potential unboundedness there is still no guarantee that $L(\theta)$ can be maximized

<u>Example</u> $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$Y_i = X_i$ rounded to the nearest integer

Suppose all $y_i = 13$