Stat 543 2-25-05

Recall : (Non-Bayesian) Optimality Thy for Point Estimation

① Rao - Blackwell

② A Lehmann- Scheffé (UMVU Estimation)

Thm In an exponential family with natural sufficient statistic (for n iid X_i's)

$$T(X) = \left(\sum T_1(X_i), \sum T_2(X_i), \ldots, \sum T_k(X_i) \right)$$

if $\mathcal{E}^* \subset \mathcal{E}$ contains an open rectangle, then if $\delta(X)$ is unbiased for $\gamma(\theta)$

and
$$\delta^*(t) = E[\delta(X) \mid T(X) = t]$$

Then $\delta^*(T(X))$ is a UMVUE of $\gamma(\theta)$

Further, if $\mathrm{Var}_\theta\, \delta^*(T(X)) < \infty \ \forall \theta$ then $\delta^*(T(X))$ is unique. (If $\delta'(X)$ is another UMVUE of $\gamma(\theta)$
$$P_\theta[\delta^*(T(X)) = \delta'(X)] = \ \forall \theta .)$$

exp family with T natural sufficient stat and non-empty interior for parameter space $\Longrightarrow$ completeness of a sufficient T $\xRightarrow[\text{Thm}]{\text{Bahadur's}}$ minimality of T

$\Downarrow$ ← Lehmann-Scheffé

Rao-Blackwellization with T produces UMVUE

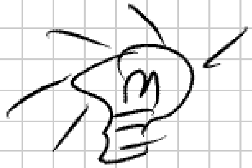
The only widely applicable method for showing completeness is to appeal to a theorem about exponential families

There are 2 ways to use this

1) If I simply recognize a function of $T(x)$ as unbiased for $\gamma(\theta)$, it must be UMVU
2) If I have an unbiased $\delta(X)$ and can see how to do the details of Rao-Blackwellization I have a construction for UMVUE

Example $X_1, X_2, \ldots, X_n$ iid Poisson(λ) $\lambda > 0$

$\gamma(\lambda) = \lambda$ – This is an exponential family with non-empty interior in natural parameter space – The natural sufficient statistic here is $T(X) = \sum_{i=1}^{n} X_i$



$\bar{X} = \frac{1}{n} T(X)$ is unbiased for λ

and therefore UMVU

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid $N(\mu, 1)$

$$\gamma(\mu) = P_\mu [X_1 < c] = \Phi(c - \mu)$$

I need an unbiased estimator to Rao-Blackwellize

$$\delta(X) = I[X_1 < c]$$

is unbiased for $\gamma(\mu)$

$T(X) = \sum X_i$ is the natural sufficient statistic

so $E[\delta(X) \mid T(X)]$ is the UMVUE it's slightly more convenient (but equivalent) to work with $T'(X) = \bar{X} = \frac{1}{n} T(X)$

$$E\left[I[X_1 < c] \mid \bar{X} \right] = P\left[X_1 < c \mid \bar{X} \right]$$

and to get this, I need the conditional of X_1 given $\bar{X}$

$$\begin{pmatrix} X_1 \\ \bar{X} \end{pmatrix} \sim \text{MVN}_2 \left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} \end{pmatrix} \right)$$

now use MVN facts to find that given $\bar{X}$, $X_1 \sim N\left(\bar{X}, \frac{n-1}{n} \right)$ so

$$P[X_1 < c \mid \bar{X}] = \Phi\left(\frac{c - \bar{X}}{\sqrt{\frac{n-1}{n}}}\right)$$

is the UMVUE of $\gamma(\mu)$

Lest you get too enthusiastic

- you can't do the Rao-Blackwellization all that often
- unbiasedness is often too much of a restriction

Example $X \sim \text{Poisson}(\lambda)$

$$\gamma(\lambda) = e^{-2\lambda} = P\left[\begin{array}{l}\text{2 independent Poisson}(\lambda) \\ \text{r.v.'s are both } 0\end{array}\right]$$

X is the natural sufficient statistic in this very simple exponential family, so any unbiased function of X is UMVU

You can check that

$$\delta(X) = (-1)^X = \begin{cases} -1 & \text{if } X \text{ is odd} \\ 1 & \text{if } X \text{ is even} \end{cases}$$

is unbiased for $\gamma(\lambda)$ and thus UMVU (it is the <u>only</u> unbiased estimator of $\gamma(\lambda)$)

RUBE's ☺

(3) The 3rd piece of optimality theory for point estimation concerns "information inequalities" and in particular, the Cramér – Rao inequality – it involves "Fisher Information" that is useful not only here but in large sample theory and applied statistics

Fisher Information — in class suppose $\Theta \subset \mathbb{R}^1$ – see handout for the (important) $\Theta \subset \mathbb{R}^k$