Stat 543 2-23-05

Recall:

<u>Rao - Blackwell Thm</u> Suppose that $\delta(X)$ is an estimator of $\gamma(\theta) \in \mathbb{R}^1$ with $E_\theta |\delta(X)| < \infty \ \forall \theta$ and that $T(X)$ is sufficient for θ. Let

$$\delta^*(t) = E[\delta(X) \mid T(X) = t]$$

Then $\delta^*(T(X))$ is an estimator of $\gamma(\theta)$ with

$$MSE_\theta(\delta^*(T(X))) \le MSE_\theta(\delta(X)) \quad \forall \theta$$

$$\left\{ \begin{array}{l} \text{and} \quad MSE_\theta\, \delta^*(T(X)) < MSE_\theta(\delta(X)) \text{ for any } \theta \\ \text{for which} \quad E_\theta(\delta(X))^2 < \infty \text{ and } P_\theta[\delta(X) \neq \delta^*(T(X))] > 0 \end{array} \right.$$

Pf: Note that $T(X)$ sufficient implies that

$$\delta^*(t) = E_{\cancel{\theta}}[\delta(X) \mid T(X) = t]$$

doesn't depend on θ so the prescription

$$\delta^*(T(X))$$

is free of θ and therefore produces an estimator (my definition of a new estimator is sensible)

$$\underset{\substack{\parallel \\ MSE_\theta(\delta(X))}}{E_\theta(\delta(X) - \gamma(\theta))^2} = E_\theta \underbrace{E_{\cancel{\theta}}[(\delta(X) - \gamma(\theta))^2 \mid T(X)]}_{(IV)}$$

$$\left(\underbrace{E[\delta(X) \mid T(X)]}_{\delta^*(T(X))} - \gamma(\theta)\right)^2$$

Jensen

$g(\cdot) = (\cdot - \gamma(\theta))^2$ is convex in $\cdot$

$$\geq E_\theta \left(\delta^*(T(X)) - \gamma(\theta) \right)^2$$

$$= MSE_\theta \left(\delta^*(T(X)) \right)$$

which is 1st part of the theorem

$\{ \quad \}$ follows from the fact $g(\cdot) = (\cdot - \gamma(\theta))^2$ is strictly convex $\blacksquare$

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid $N(\theta, 1)$

Consider $\gamma(\theta) = E_\theta X_1^2 = \theta^2 + 1$

$T(X) = \sum X_i$ is sufficient for θ

a method of moments estimator of $\gamma(\theta)$ is

$$\frac{1}{n}\sum X_i^2 = \frac{1}{n}\sum (X_i - \bar{X})^2 + \bar{X}^2$$

What is the "Rao-Blackwellization" of this?

$$E\left[\frac{1}{n}\sum X_i^2 \,\middle|\, \sum X_i\right] = \frac{1}{n} E\left[\sum (X_i - \bar{X})^2 \,\middle|\, \sum X_i\right] + E\left[\bar{X}^2 \,\middle|\, \sum X_i\right]$$

independent

$$= \frac{1}{n}\left(\begin{matrix}\text{mean of}\\ \text{a } \chi^2_{n-1}\end{matrix}\right) + \bar{X}^2$$

$$= \frac{n-1}{n} + \bar{X}^2$$

$$= 1 + \bar{X}^2 - \frac{1}{n}$$

This estimator is silly because $\bar{X}^2 - \frac{1}{n}$ can be negative and I know $\gamma(\theta) \geq 1$

an improvement on this is

$$\delta^{**}(X) = \begin{cases} 1 + \bar{X}^2 - \frac{1}{n} & \text{if } \bar{X}^2 > \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

but the basic point stands: Rao-Blackwell promises improvement, it doesn't promise that when you're done you have something that is sensible

---

This business of Rao-Blackwellization can sometimes be used to produce "best" estimators

δ "best" means that for any other estimator δ'

$$MSE_\theta(\delta(x)) \le MSE_\theta(\delta'(x)) \quad \underline{\underline{\forall\ \theta}}$$

This is just too much to hope for if I consider all possible estimators

<u>Example</u> $X \sim N(\theta, 1)$ $\gamma(\theta) = \theta$

$\delta'(x) = 13$ has 0 MSE/risk if $\theta = 13$

$$MSE(\delta'(x)) = \underbrace{(13-\theta)^2}_{bias^2(\delta'(x))}$$

$\delta''(x) = 75$ has 0 MSE/risk if $\theta = 75$

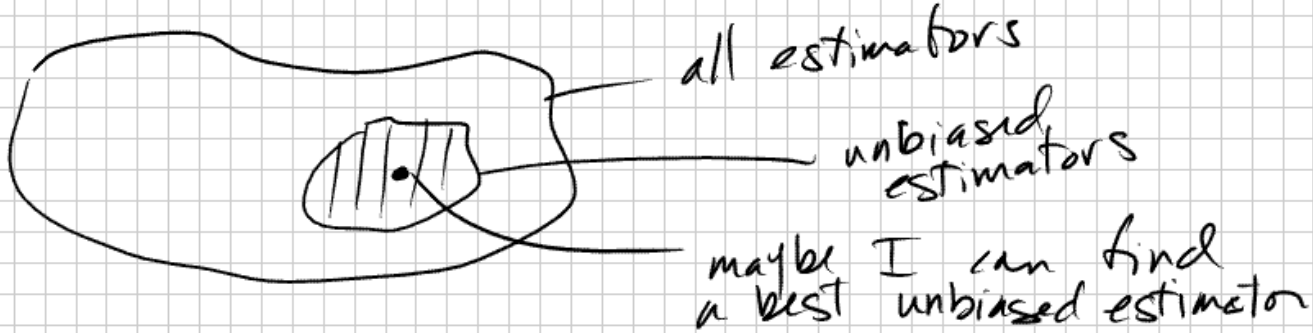
??? an overall best estimator must have 0 MSE $\forall\ \theta$

In order to find a "best" estimator I'm going to have to limit the set of estimators I'm willing to consider to some "sensible" class of estimators

— one standard thing to do is to restrict attention to "unbiased" estimators

<u>Def</u> $\delta(X)$ is an unbiased estimator of $\gamma(\theta)$ if

$$E_\theta\, \delta(X) = \gamma(\theta) \qquad \forall\, \theta$$



note that for unbiased estimators

$$MSE_\theta(\delta(X)) = Var_\theta\, \delta(X)$$

and a "best unbiased estimator" is a "uniformly minimum variance unbiased estimator"