Stat 543 2-2-05

Estimation X $\delta(X)$ $\gamma(\theta)$

Example (not so standard) (C+B)

$X_1, X_2, \ldots, X_n$ iid Binomial (k, p) with both k, p unknown ... how to estimate $\gamma(k,p) = (k,p)$

There are several "standard" ways of inventing estimator

MOM

Maximum Likelihood (M estimation)

Bayes

Method of Moments

Suppose a single observation takes values in $\mathbb{R}^1$ so that

$$E_\theta X^j = \mu_j(\theta)$$

makes sense — Suppose that for some (usually small) number of moments, r, we can write

$$\gamma(\theta) = h\left(\mu_1(\theta), \mu_2(\theta), \ldots, \mu_r(\theta)\right)$$

Then where entries of $X=(X_1,\ldots,X_n)$ are iid $f(x|\theta)$, let

$$\hat{\mu}_{jn} = \frac{1}{n}\sum_{i=1}^{n} X_i^j$$

And a plausible estimator of $\gamma(\theta)$ is then

$$\delta_n(X) = h(\hat{\mu}_{1n}, \hat{\mu}_{2n}, \ldots, \hat{\mu}_{rn})$$

Example CB $X_1,\ldots,X_n$ iid $Bi(k,p)$

$$\mu_1(k,p) = kp$$

$$\mu_2(k,p) = kp(1-p)+(kp)^2$$

$$= kp - kp^2 + k^2p^2$$

So $k = \frac{\mu_1}{p}$ and $\mu_2 = \mu_1 - \mu_1 p + \mu_1^2$

i.e. $p = \mu_1 + 1 - \frac{\mu_2}{\mu_1}$

and $k = \frac{\mu_1}{\mu_1 + 1 - \frac{\mu_2}{\mu_1}}$

And this suggests that to estimate (k, p) I might use

$$\delta_n(X) = \left(\frac{\hat{\mu}_{1n}}{\hat{\mu}_{1n} + 1 - \frac{\hat{\mu}_{2n}}{\hat{\mu}_{1n}}} \, , \; \hat{\mu}_{1n} + 1 - \frac{\hat{\mu}_{2n}}{\hat{\mu}_{1n}} \right)$$

note this isn't necessarily an integer ☹

nevertheless this kind of thing makes some sense – e.g.

Provided $E|X|^r < \infty$ the LLN says that

$$\hat{\mu}_{sn} \xrightarrow{P_\theta} \mu_s(\theta) \quad \forall\, s \le r$$

So if $h(\cdot,\cdot,\cdots,\cdot)$ is cont$\underline{s}$ at

$$(\mu_1(\theta), \mu_2(\theta), \ldots, \mu_r(\theta))$$

then we can conclude that

$$\delta_n(X) = h(\hat{\mu}_{1n}, \ldots, \hat{\mu}_{rn}) \xrightarrow{P_\theta} \gamma(\theta)$$

this is an important one ... it is called "consistency"

Def If $\{\delta_n(X)\}$ is a sequence of estimators of $\gamma(\theta)$ and

$$\delta_n(X) \xrightarrow{P_{\theta_0}} \gamma(\theta_0)$$

we'll say that the sequence is consistent for $\gamma(\theta)$ at $\theta = \theta_0$

$\cancel{\delta_n(X^n)}$

Example C+B $X_1, \ldots, X_n$ iid $Bi(k,p)$

$$h_2(m_1, m_2) = m_1 + 1 - \frac{m_2}{m_1}$$

$$h_1(m_1, m_2) = \frac{m_1}{h_2(m_1, m_2)}$$

cont$^{\underline{s}}$ on $(0,\infty)^2$

Binomial dsns have 2nd moments so that as long as $p > 0$ and $k \geq 1$

$$\hat{\mu}_{1n} \xrightarrow{P_{k,p}} \mu_1 \quad \text{and} \quad \hat{\mu}_{2n} \xrightarrow{P_{k,p}} \mu_2$$

and thus $\delta_n(X)$ is consistent for (k, p) for all such (k, p)

<u>Example</u> Truncated Poisson — marginal pmf

$$f(x \mid \lambda) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{\lambda^x}{x!} \qquad x = 1, 2, \ldots$$

$$\mu_1(\lambda) = E_\lambda X = \frac{\lambda}{1 - e^{-\lambda}}$$

Suppose $X_1, \ldots, X_n$ are iid with this marginal pmf

$$\hat{\mu}_{1n} = \bar{X} = \frac{1}{n}\sum X_i$$

use this to estimate λ

$$\mu_1(\lambda) = \frac{\lambda}{1-e^{-\lambda}}$$

??"solve for λ in terms of μ_1" ???

But a MOM estimator of λ is a solution to

$$\bar{X} = \frac{\lambda}{1-e^{-\lambda}}$$

(I have to do a little numerical analysis)

an "estimating equation" for λ

Often MoM estimators are sensible (at least after some "adjustment") but not always "optimal" — "usually" maximum likelihood does better

$L(\theta)$