Stat 543 2-16-05

Recall: Decision-Theoretic argument for the conditional/posterior mean as a (Bayes) estimator

To minimize $R(G, \delta)$ it suffices to for each x minimize

$$\int L(\theta, a)\, g(\theta \mid x)\, d\theta = \text{conditional expected loss for action } a \text{ given } X = x$$

In the case that $L(\theta, a) = (\gamma(\theta) - a)^2$ this is

"minimize

$$\int (\gamma(\theta) - a)^2\, g(\theta \mid x)\, d\theta$$

by choice of a"

i.e. minimize

$$\int \gamma^2(\theta)\, g(\theta|x)\, d\theta \;-\; 2a \int \gamma(\theta)\, g(\theta|x)\, d\theta \;+\; \int a^2 g(\theta|x)\, d\theta$$

$$\int a^2 g(\theta|x)\, d\theta = a^2$$

This is a quadratic in a with minimum where

$$-2\int \gamma(\theta)\, g(\theta|x)\, d\theta + 2a = 0$$

i.e.

$$a = \int \gamma(\theta)\, g(\theta|x)\, d\theta$$

i.e. $\delta(x) = \int \gamma(\theta)\, g(\theta|x)\, d\theta$ gives the Bayes estimator $\delta(X)$

BTW if I start with a different loss function I get a different prescription for a Bayes estimator

<u>Example</u> $\theta \in \mathbb{R}^1$ $L(\theta, a) = |\theta - a|$

(absolute error loss)

$$\int |\theta - a| \, g(\theta|x) \, d\theta$$

is minimized by any a that is a median of the posterior i.e. if

$$\delta(x) = \text{median of } G(\theta|x)$$

$\delta(X)$ is a Bayes estimator of θ

How does one implement the Bayes prescription in a real/complicated problem?

Suppose $\theta \in \mathbb{R}^k$

If $\underbrace{k \text{ is small}}_{1 \text{ or } 2}$ $\Longrightarrow$ I can understand

$$g(\theta|x) \propto L(\theta)\, g(\theta)$$

pretty easily by plotting and low-dimensional numerical integration

If k is big I have problems ... I may not even be able to compute

$$\int L(\theta)\, g(\theta)\, d\theta$$

using numerical analysis

how then to compute things like posterior mean θ_i given $X = x$

$$\frac{\int \theta_i \, L(\theta)\, g(\theta)\, d\theta}{\int L(\theta)\, g(\theta)\, d\theta}$$

perhaps one can use simulation

Suppose we're interested in

$$Q = \int q(\theta)\, g(\theta | x)\, d\theta$$

for some $q : R^k \longrightarrow R^1$ ... q could be $q(\theta) = \theta_{11}$

or $\quad q(\theta) = I[\theta \in \Delta]$ for $\Delta \subset \mathbb{R}^k$

If I could generate/simulate $\theta_1^*, \theta_2^*, \ldots$ iid $G(\theta|x)$ I might approximate Q by by

$$\frac{1}{n}\sum_{i=1}^{n} q(\theta_i^*)$$

figuring that LLN says that

$$\frac{1}{n}\sum_{i=1}^{n} q(\theta_i^*) \xrightarrow{P} Q$$

$G(\theta|x)$ probability

At first look, this may be no help, because it's not obvious (particularly if I can't

compute $\int L(\theta)\,g(\theta)\,d\theta$) that I'm going to be able to do the simulation — but then the situation is not so bleak — there are 1) the rejection algorithm
2) Markov Chain Monte Carlo

Rejection Algorithm (for sampling from $g(\theta|x)$ when I only know $L(\theta)g(\theta)$)

Suppose that $h(\theta)$ specifies a dsn from which I can easily sample and $\exists$ a constant $M>0$ such that

$$Mh(\theta) > \underbrace{L(\theta)\,g(\theta)}_{\text{proportional to } g(\theta|x)}$$

To generate $\theta^* \sim g(\theta|x)$ I can

1. generate $\theta^{**} \sim h(\theta)$
2. generate $u \sim (0,1)$
3. if $Muh(\theta^{**}) < L(\theta^{**})\,g(\theta^{**})$

   set $\theta^* = \theta^{**}$

   otherwise return to 1

And generating $\theta_1^*, \theta_2^*, \ldots$ in this way I can approximate Q

"Argument" for why the rejection algorithm works — Note that

$$P[\text{algorithm stops at current iteration} \mid \theta^{**}] = \frac{L(\theta^{**})\, g(\theta^{**})}{M\, h(\theta^{**})}$$

So $P[$algorithm fails to stop on a particular iteration | it gets to that iteration$]$

$$= 1 - \int \frac{L(\theta)\, g(\theta)}{M\, h(\theta)}\, h(\theta)\, d\theta$$

$$= 1 - \frac{\int L(\theta)\, g(\theta)\, d\theta}{M}$$

$$P[\text{ iterations are required and } \underbrace{\theta^* \text{ is near } \theta}_{\text{in "}\Delta\theta\text{"}}]$$

$$\approx \left(1 - \frac{\int L(\theta)g(\theta)\,d\theta}{M}\right)^{i-1} \left(\frac{L(\theta)g(\theta)}{M h(\theta)}\right)$$

$$\times\, h(\theta)\left(\text{vol "}\Delta\theta\text{"}\right)$$