Stat 543 1-31-05

Notational Matter :

$$f(x|\eta) = \cancel{A(\eta)}\, h(x) \exp \sum_{j=1}^{k} \eta_j T_j(x)$$

$C(\eta)$

Recall Example $N(\mu, \sigma^2)$

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\eta_1 x^2 + \eta_2 x\right) \exp\left(-\frac{\mu^2}{2\sigma^2}\right)$$

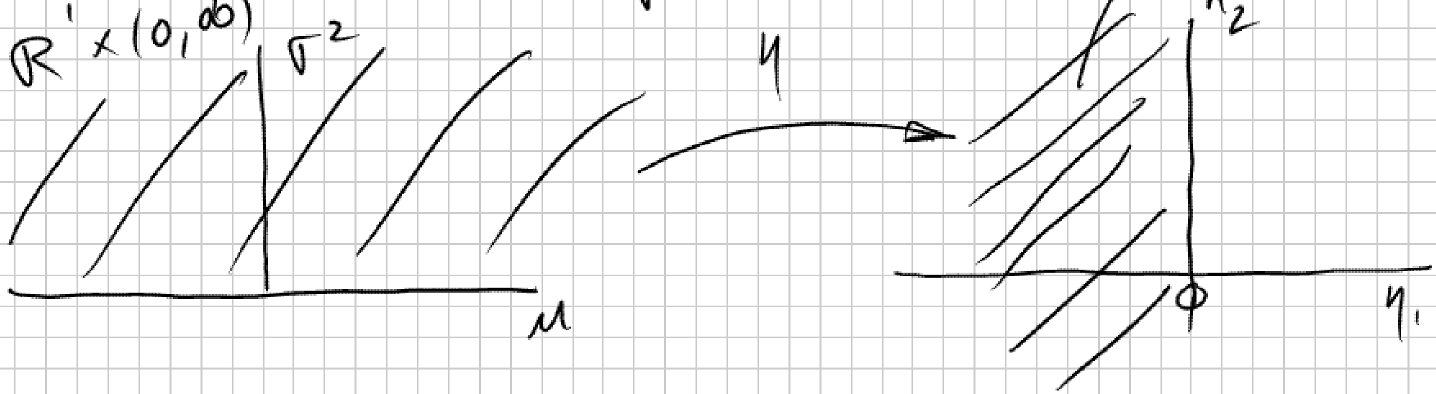
$$T_1(x) = x^2 \quad T_2(x) = x \quad \eta_1 = -\frac{1}{2\sigma^2} \quad \eta_2 = \frac{\mu}{\sigma^2}$$

We can thus see that for $X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$ the natural sufficient statistic is $T(X) = \left(\sum X_i^2, \sum X_i \right)$ and we can apply the exponential family stuff to conclude that $T(X)$ is minimal sufficient

$$\eta(\mu, \sigma^2) = \left(\eta_1(\mu, \sigma^2), \eta_2(\mu, \sigma^2) \right)$$

where $\eta_1(\mu, \sigma^2) = -\frac{1}{2\sigma^2}$ $\quad \eta_2 = \frac{\mu}{\sigma^2}$

$\Theta = \mathbb{R}^1 \times (0, \infty)$



$\mathcal{E}_\Theta$ is a half plane and contains an open set — one thing this implies is that $T(X)$ is minimal sufficient

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??? What does this requirement for $\mathcal{E}_\Theta$ to contain an open set intend to exclude???
To think about this, consider two sub-models of the $N(\mu, \sigma^2)$ model ...

1) $N(\mu, 1)$
2) $N(\mu, \mu^2)$

Example $N(\mu,1)$

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2 + \overset{T(x)}{x}\,\underset{\eta}{\mu} - \frac{\mu^2}{2}\right)$$

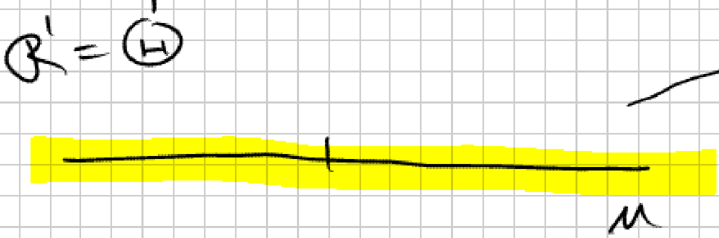
with $\eta = \mu$ we have a 1-dimensional exponential family ... for iid observations the natural sufficient statistic is

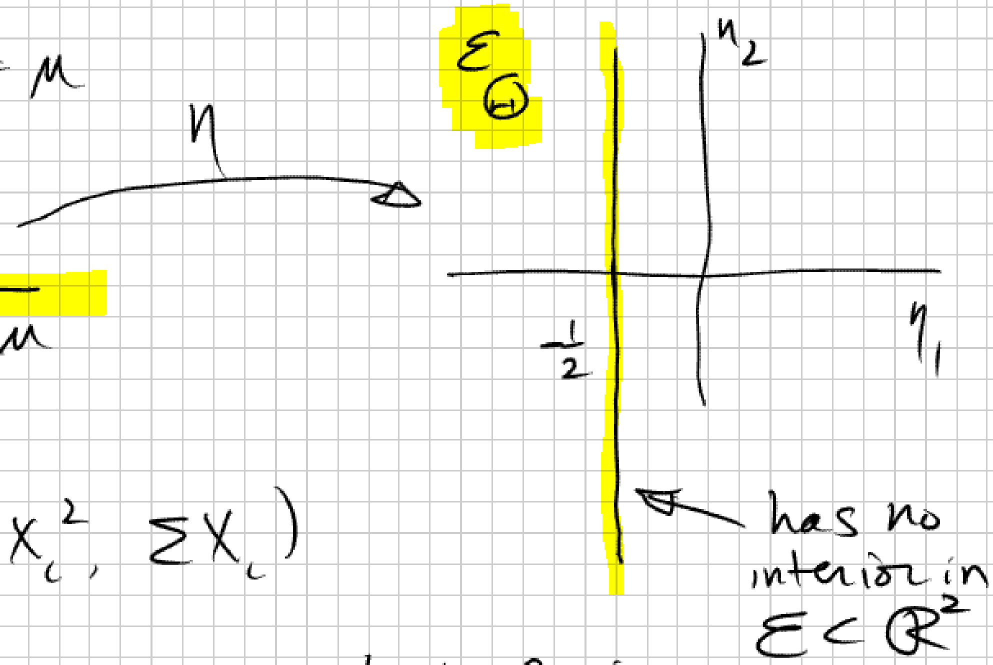
$$T(X) = \sum X_i \qquad \Re^1$$

and since Γ (or Θ) contains an open rectangle this is minimal sufficient

Relation to full $N(\mu,\sigma^2)$ model?

$\eta_1 = -\frac{1}{2} \qquad \eta_2 = \mu$



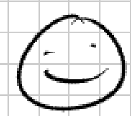


$X_1, \ldots, X_n$ iid

$$T(X) = \left(\sum X_i^2, \sum X_i\right)$$

sufficient

I can't use the exponential family machinery to conclude $\left(\sum X_i^2, \sum X_i\right)$ is minimal sufficient —



<u>Example</u> $N(\mu, \mu^2)$

$\eta_1(\mu)$ $\eta_2(\mu$

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}\,|\mu|} \exp\left(-\frac{1}{2\mu^2}x^2 + \frac{x}{\mu} - \frac{1}{2}\right)$$

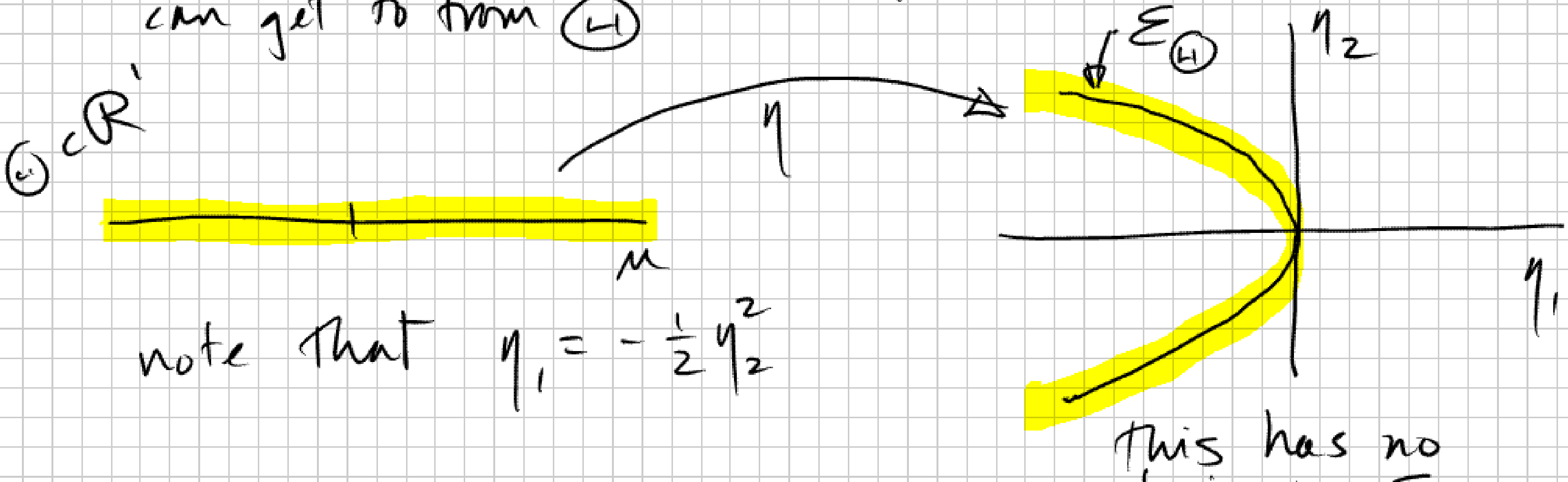
with $\eta_1(\mu) = -\dfrac{1}{2\mu^2}$ $\quad \eta_2 = \dfrac{1}{\mu}$

$T_1(x) = x^2 \quad T_2(x) = x$

this is some sort of exponential family with $X_1, \ldots, X_n$ the natural sufficient statistic is

$$T(X) = \left(\sum X_i^2, \sum X_i\right)$$

But .... (4) is really only "1-dimensional" and so also is the set $\eta = (\eta_1, \eta_2)$ that I can get to from (4)



note that $\eta_1 = -\frac{1}{2}\eta_2^2$

?? Is $\left(\sum X_i^2, \sum X_i\right)$ minimal sufficient

This is an example of a "curved exponential family" – ∃ Theory for such families

Begin reading Ch 2 – The topic is "Estimation"

The basic idea is to use X to guess at the value of θ or some function of θ, say $\gamma(\theta) \in \mathbb{R}^k$

__Def__ A statistic $\delta(X)$ taking values in $\gamma(\Theta)$ (range of γ) is called a point estimator of $\gamma(\theta)$

__Standard Examples__

$X_1, X_2, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$\delta(X) = (\bar{X}, s^2)$ is a obvious point estimator of (μ, σ^2)

(? What about $\gamma(\mu, \sigma^2) = \Phi\left(\frac{17-\mu}{\sigma}\right)$??
How to guess at this?)

② $X_1, X_2, \ldots, X_n$ iid Bernoulli p

$\delta(X) = \frac{\sum x_i}{n}$ is an obvious point estimator of p

(? What about, $\gamma(p) = p^2$? How to approximate?)