Stat 543 1-21-05

Recall:

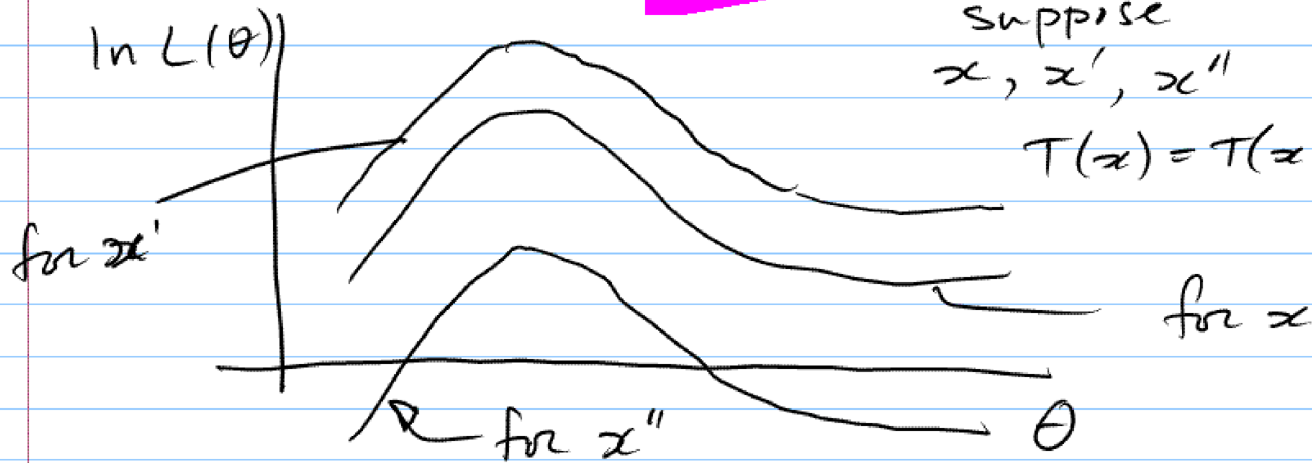
Thm (FNHS Factorization)

$T(X)$ sufficient $\Longleftrightarrow$ $\underbrace{f(x|\theta) = g(T(x), \theta)\, h(x)}$

"the shape of the loglikelihood depends on x only through $T(x)$"

$T(X)$ sufficient $\Longleftrightarrow$ $L(\theta) = g(T(x), \theta)\, h(x)$

$$\ln L(\theta) = \ln g(T(x), \theta) + \ln h(x)$$



suppose x, x', x''

$T(x) = T(x') = T(x'')$

Example $X_1, \ldots, X_n$ iid Poisson(λ)

$\lambda \geq 0$

$$f(x \mid \lambda) = \begin{cases} \prod_{i=1}^{n} \left(\dfrac{e^{-\lambda} \lambda^{x_i}}{x_i!} \right) & \text{all } x_i \text{ are nonnegative integers} \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod x_i!} \, I\left[\text{all } x_i \text{ are nonnegative integers}\right]$$

Let $g(t,\lambda) = e^{-n\lambda}\lambda^{t}$

$$h(x) = \frac{1}{\prod x_i!} I\left[\text{all } x_i \text{ are nonnegative integers}\right]$$

$$T(x) = \sum x_i$$

we can then write

$$f(x|\lambda) = g(T(x),\lambda)\,h(x)$$

So $T(X) = \sum X_i$ is sufficient for λ

<u>Example</u> $X_1, X_2, \ldots, X_n$ iid Beta (α, β)

$$f(x \mid \alpha, \beta) = \left(\prod_{i=1}^{n} \frac{x_i^{\alpha-1} (1-x_i)^{\beta-1}}{B(\alpha, \beta)} \right) I\left[\begin{matrix}\text{all } x_i \text{ are} \\ \text{in } (0,1)\end{matrix}\right]$$

$$= \left(\frac{1}{B(\alpha, \beta)} \right)^n \left(\prod x_i \right)^{\alpha-1} \left(\prod (1-x_i) \right)^{\beta-1} I\left[\quad \right]$$

Let $g(t_1, t_2, \alpha, \beta) = \left(\frac{1}{B(\alpha, \beta)} \right)^n t_1^{\alpha-1} t_2^{\beta-1}$

$$h(x) = I\left[\begin{matrix}\text{all } x_i \text{ are} \\ \text{in } (0,1)\end{matrix}\right]$$

$$T(x) = \left(\prod x_i, \prod (1-x_i) \right)$$

you can check that

$$f(x|\alpha,\beta) = g(T(x),\alpha,\beta)\, h(x) \quad \text{so}$$

that by FNHS

$$T(X) = \left(\prod X_i , \prod (1-X_i) \right)$$

is sufficient for (α,β)

<u>Proof</u> of FNHS in discrete case:

Suppose that $T(X)$ is sufficient for θ
and let $\theta_0 \in \Theta$
For any t and θ with

$$P_\theta[T(X)=t] > 0$$

Sufficiency says that

$$P_\theta[X=x \mid T(X)=t] = P_{\theta_0}[X=x \mid T(X)=t]$$

So

$$f(x|\theta) = P_\theta[X=x] = P_\theta[X=x \text{ and } T(X)=T(x)]$$

$$= \underbrace{P_\theta[X=x \mid T(X)=T(x)]}_{\underbrace{P_{\theta_0}[X=x \mid T(X)=T(x)]}_{h(x)}} \underbrace{P_\theta[T(X)=T(x)]}_{g(T(x),\,\theta)}$$

where $g(t,\theta) = P_\theta[T(X)=t]$

On the other hand, suppose $\exists$ nonnegative $g(t,\theta)$ and $h(x)$ so that

$$f(x|\theta) = g(T(x),\theta)\, h(x)$$

Then

$$P_\theta\left[T(X)=t\right] = \sum_{\substack{x \text{ s.t.} \\ T(x)=t}} g(T(x),\theta)\, h(x)$$

$$= g(t,\theta) \sum_{\substack{x \text{ s.t.} \\ T(x)=t}} h(x)$$

So for any t with $P_\theta[T(X)=t]>0$

$$P_\theta[X=x \mid T(X)=t] = \frac{P_\theta[X=x \text{ and } T(X)=t]}{P_\theta[T(X)=t]}$$

$$= I[T(x)=t]\,\frac{\cancel{g(t,\theta)}\,h(x)}{\cancel{g(t,\theta)}\displaystyle\sum_{\substack{x \text{ s.t.}\\ T(x)=t}} h(x)}$$

$$= I[T(x)=t]\,\frac{h(x)}{\displaystyle\sum_{\substack{x \text{ s.t.}\\ T(x)=t}} h(x)}$$

and this doesn't involve θ, i.e. $T(X)$ is sufficient for θ ∎

<u>Example</u> $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$\mu \in \mathbb{R}$, $\sigma^2 > 0$

$$T(X) = \left(\sum X_i, \sum X_i^2 \right) = (T_1(X), T_2(X))$$

("equivalent to" $\bar{X}, s^2$)

$$f(x|\mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}\left(\sum(x_i-\mu)^2\right)\right]$$

$$= \left(\quad\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\mu\sum x_i + n\mu^2\right)\right]$$

So with

$$g(t,\mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}\left(t_1 - 2\mu t_2 + n\mu^2\right)\right]$$

$$h(x) \equiv 1$$

I apply Factorization Thm and $T(X)$ is sufficient for (μ,σ^2)