STAT 543 Homework 3 Solution

1.

Problem 1.5.11

Let $X_1, X_2, \ldots, X_n$ be a sample form the uniform, $U(0,\theta)$ distribution. Show that $X_{(n)} = \max\left\{X_i; 1 \le i \le n\right\}$ is minimal sufficient for θ.

Solution:

Since $X_1, X_2 \ldots, X_n$ be a sample from the uniform, $U(0,\theta)$ distribution, then

$$f(\underline{x}\,|\,\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 < x_i < \theta) = \frac{1}{\theta^n} I(0 < x_{(1)}) I(x_{(n)} < \theta).$$

Now, suppose $\exists k(\underline{x},\underline{y}) > 0$, $s.t.$

$$f(\underline{y}\,|\,\theta) = k(\underline{x},\underline{y}) f(\underline{x}\,|\,\theta)$$

holds for $\forall\theta$;

i.e.

$$\frac{1}{\theta^n} I(0 < y_{(1)}) I(y_{(n)} < \theta) = k(x,y) \frac{1}{\theta^n} I(0 < x_{(1)}) I(x_{(n)} < \theta)$$

holds for $\forall\theta$.

The above equality requires that $x_{(n)} = y_{(n)}$ .Otherwise, if $x_{(n)} < y_{(n)}$, then for $\theta \in (x_{(n)}, y_{(n)})$ and $x_i > 0 \,\&\, y_i > 0, i = 1,\ldots,n$, $LHS = 0$, but $RHS = k(x,y)\frac{1}{\theta^n} > 0$; And if $x_{(n)} > y_{(n)}$, then when $\theta \in (x_{(n)}, y_{(n)})$ and $x_i > 0 \,\&\, y_i > 0, i = 1,\ldots,n$, $LHS = \frac{1}{\theta^n} > 0$, but $RHS = 0$.

Let $T(\underline{x}) = x_{(n)}$, then the equality forces that $T(\underline{x}) = T(\underline{y})$. So, $T(\underline{x}) = x_{(n)}$ is minimal sufficient.

Problem 1.5.16

Let $X_1, X_2, \ldots, X_m$; $Y_1, Y_2, \ldots, Y_n$, be independently distributed according to $\mathcal{N}(\mu,\sigma^2)$ and $N(\eta,\tau^2)$, respectively. Find minimal sufficient statistics for the following three cases:

(i) μ,η,σ,τ are arbitrary: $-\infty < \mu,\eta < \infty, 0 < \sigma,\tau$.

(ii) $\sigma=\tau$ and μ,η,σ are arbitrary.

(iii) $\mu=\eta$ and μ,σ,τ are arbitrary.

Solution:

Since $X_1,\ldots,X_m \sim iidN(\mu,\sigma^2)$, $N(\eta,\tau^2)$, and $X_1,\ldots,X_m \,\&\, Y_1,\ldots,Y_n$ are all independent, then

$$f(x_1,\ldots,x_m,y_1,\ldots,y_n)=(2\pi)^{-\frac{m+n}{2}}(\sigma^2)^{-\frac{m}{2}}(\tau^2)^{-\frac{n}{2}}\exp\{-\frac{\sum_{i=1}^m(x_i-\mu)^2}{2\sigma^2}-\frac{\sum_{i=1}^n(y_i-\mu)^2}{2\tau^2}\}$$

$$=(2\pi)^{-\frac{m+n}{2}}\exp\{-\frac{m\mu^2}{2\sigma^2}-\frac{n\mu^2}{2\tau^2}\}\cdot\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^m x_i^2+\frac{\mu}{\sigma^2}\sum_{i=1}^m x_i-\frac{1}{2\tau^2}\sum_{i=1}^n y_i^2+\frac{\mu}{\tau^2}\sum_{i=1}^n y_i\}$$

(i)

Let $\tilde{\eta}=(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2},-\frac{1}{2\tau^2},\frac{\eta}{\tau^2})'$, and $T(\underline{x})=(\sum_{i=1}^m x_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i^2,\sum_{i=1}^n y_i)'$, then by the property of exponential family, $T(\underline{x})=(\sum_{i=1}^m x_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i^2,\sum_{i=1}^n y_i)'$ is the natural sufficient statistics for $\tilde{\eta}=(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2},-\frac{1}{2\tau^2},\frac{\eta}{\tau^2})'$.

Since $-\infty<\mu,\eta<\infty$ and $\sigma,\tau>0$, then $0<\frac{1}{\sigma^2},\frac{1}{\tau^2}<\infty$, and $-\infty<\frac{\mu}{\sigma^2},\frac{\eta}{\tau^2}<\infty$, and further $-\infty<-\frac{1}{2\sigma^2},-\frac{1}{2\tau^2}<0$. We can see that the natural parameter space Σ of η contains open rectangles, which implies that $T(\underline{x})=(\sum_{i=1}^m x_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i^2,\sum_{i=1}^n y_i)'$ is also minimal sufficient.

(ii)

If $\sigma=\tau$, then $\tilde{\eta}=(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2},\frac{\eta}{\sigma^2})'$. Let $T(\underline{x})=(\sum_{i=1}^m x_i^2+\sum_{i=1}^n y_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i)'$, then by the property of exponential family, $T(\underline{x})=(\sum_{i=1}^m x_i^2+\sum_{i=1}^n y_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i)'$ is the natural sufficient statistics for $\tilde{\eta}=(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2},\frac{\eta}{\sigma^2})'$.

Since $-\infty<\mu,\eta<\infty$ and $\sigma>0$, then $0<\frac{1}{\sigma^2}<\infty$, and $-\infty<\frac{\mu}{\sigma^2},\frac{\eta}{\sigma^2}<\infty$, and further $-\infty<-\frac{1}{2\sigma^2}<0$. In this case the natural parameter space Σ of η contains open rectangles, which implies that $T(\underline{x})=(\sum_{i=1}^m x_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i^2,\sum_{i=1}^n y_i)'$ is also minimal sufficient.

(iii)

If $\mu=\eta$, then $\tilde{\eta}=(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2},-\frac{1}{2\tau^2},\frac{\mu}{\tau^2})'$. Let $T(\underline{x})=(\sum_{i=1}^m x_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i^2,\sum_{i=1}^n y_i)'$, then by the property of exponential family, $T(\underline{x})=(\sum_{i=1}^m x_i^2,\sum_{i=1}^m x_i,\sum_{i=1}^n y_i^2,\sum_{i=1}^n y_i)'$ is the natural sufficient statistics for $\tilde{\eta}=(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2},-\frac{1}{2\tau^2},\frac{\mu}{\tau^2})'$.

Without loss of generality, for $\underline{x_1}=(x_{11},\ldots,x_{1m},y_{11},\ldots,y_{1n})'$ and $\underline{x_2}=(x_{21},\ldots,x_{2m},y_{21},\ldots,y_{2n})'$, assume $\sum_{i=1}^m x_{1i}^2\ \ge\sum_{i=1}^m x_{2i}^2$, $\sum_{i=1}^n y_{1i}^2\ \ge\sum_{i=1}^n y_{2i}^2$, $\sum_{i=1}^m x_{1i}\ \ge\sum_{i=1}^m x_{2i}$ and $\sum_{i=1}^n y_{1i}\ \ge\sum_{i=1}^n y_{2i}$.

Now suppose $\exists\ \ k(x_1,x_2)>0$ s.t.

$$f(x_1 \mid \tilde{\eta}) = f(x_2 \mid \tilde{\eta})\,k(x_1, x_2)$$

hold for $\forall \quad \tilde{\eta} = \left(\frac{1}{\sigma^2}, \frac{1}{\tau^2}, \frac{\mu}{\sigma^2}, \frac{\mu}{\tau^2}\right)$

, then

$$\frac{f(\underline{x}_1 \mid \tilde{\eta})}{f(\underline{x}_2 \mid \tilde{\eta})} = \exp\left(-\frac{\sum_{i=1}^{m} x_{1i}^2 - \sum_{i=1}^{m} x_{2i}^2}{2\sigma^2} - \frac{\sum_{i=1}^{n} y_{1i}^2 - \sum_{i=1}^{n} y_{2i}^2}{2\tau^2} + \frac{\mu\left(\sum_{i=1}^{m} x_{1i} - \sum_{i=1}^{m} x_{2i}\right)}{\sigma^2} + \frac{\mu\left(\sum_{i=1}^{n} y_{1i} - \sum_{i=1}^{n} y_{2i}\right)}{\tau^2}\right)$$

Let μ=0, $\sigma^2 = \dfrac{\sum_{i=1}^{n} y_{1i}^2 - \sum_{i=1}^{n} y_{2i}^2}{\sum_{i=1}^{m} x_{1i}^2 - \sum_{i=1}^{m} x_{2i}^2}$, then

$$\frac{f(\underline{x}_1 \mid \tilde{\eta})}{f(\underline{x}_2 \mid \tilde{\eta})} = \exp\left(-\left(\sum_{i=1}^{n} y_{1i}^2 - \sum_{i=1}^{n} y_{2i}^2\right)\left(1 + \frac{1}{\tau^2}\right)\right) = k(x, y)$$

holds for $\forall \tau^2 > 0$. This requires $\sum_{i=1}^{n} y_{1i}^2 = \sum_{i=1}^{n} y_{2i}^2$.

Similarly, let μ=0, $\tau^2 = \dfrac{\sum_{i=1}^{m} x_{1i}^2 - \sum_{i=1}^{m} x_{2i}^2}{\sum_{i=1}^{n} y_{1i}^2 - \sum_{i=1}^{n} y_{2i}^2}$, then

$$\frac{f(\underline{x}_1 \mid \tilde{\eta})}{f(\underline{x}_2 \mid \tilde{\eta})} = \exp\left(-\left(\sum_{i=1}^{m} x_{1i}^2 - \sum_{i=1}^{m} x_{2i}^2\right)\left(1 + \frac{1}{\sigma^2}\right)\right) = k(x, y)$$

holds for $\forall \sigma^2 > 0$. This requires $\sum_{i=1}^{m} x_{1i}^2 = \sum_{i=1}^{m} x_{2i}^2$.

Further, known that $\sum_{i=1}^{m} x_{1i}^2 = \sum_{i=1}^{m} x_{2i}^2$ and $\sum_{i=1}^{n} y_{1i}^2 = \sum_{i=1}^{n} y_{2i}^2$

, let μ=1, $\sigma^2 = \dfrac{\sum_{i=1}^{n} y_{1i} - \sum_{i=1}^{n} y_{2i}}{\sum_{i=1}^{m} x_{1i} - \sum_{i=1}^{m} x_{2i}}$, then

$$\frac{f(\underline{x}_1 \mid \tilde{\eta})}{f(\underline{x}_2 \mid \tilde{\eta})} = \exp\left(-\left(\sum_{i=1}^{n} y_{1i} - \sum_{i=1}^{n} y_{2i}\right)\left(1 + \frac{1}{\tau^2}\right)\right) = k(x, y)$$

holds for $\forall \tau^2 > 0$. This requires $\sum_{i=1}^{n} y_{1i} = \sum_{i=1}^{n} y_{2i}$.

Similarly, let μ=1, $\tau^2 = \dfrac{\sum_{i=1}^{m} x_{1i} - \sum_{i=1}^{m} x_{2i}}{\sum_{i=1}^{n} y_{1i} - \sum_{i=1}^{n} y_{2i}}$, then

$$\frac{f(\underline{x}_1 \mid \tilde{\eta})}{f(\underline{x}_2 \mid \tilde{\eta})} = \exp\left(-\left(\sum_{i=1}^{m} x_{1i} - \sum_{i=1}^{m} x_{2i}\right)\left(1+\frac{1}{\sigma^2}\right)\right) = k(x, y)$$

holds for $\forall \sigma^2 > 0$. This requires $\sum_{i=1}^{m} x_{1i} = \sum_{i=1}^{m} x_{2i}$.

From above, we see that $T(\underline{x}_1) = T(\underline{x}_2)$, and thus $T = \left(\sum_{i=1}^{n} T_1(X_i), \dots, \sum_{i=1}^{n} T_k(X_i)\right)$ is minimal sufficient.

Problem 1.6.2

Suppose $X_1, X_2, \dots, X_n$ is as in Problem *1.5.3*. In each of the cases (a), (b) and (c), show that the distribution of **X** forms a one-parameter exponential family. Identify η, B, T, and h.

Solution:

(a)

Since

$$p(x,\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \exp\{(\theta-1)\sum_{i=1}^{n} \ln x_i + n\ln\theta\},$$

let $\eta = \theta - 1$, $T(\underline{x}) = \sum_{i=1}^{n} \ln x_i$, $B(\theta) = -n\ln\theta$, and $h(\underline{x}) = 1$, then

$$p(x,\theta) = h(\underline{x})\exp\{\eta T(\underline{x}) + B(\theta)\}.$$

So, X forms a one-parameter exponential family.

(b)

Since

$$p(x,\theta) = \theta^n a^n \prod_{i=1}^{n} x_i^{a-1}\exp(-\theta x^a) = a^n \prod_{i=1}^{n} x_i^{a-1}\cdot\exp\{-\theta\sum_{i=1}^{n} x_i^{a} + n\ln\theta\},$$

let $\eta = -\theta$, $T(\underline{x}) = \sum_{i=1}^{n} x_i^{a}$, $B(\theta) = -n\ln\theta$, and $h(\underline{x}) = a^n \prod_{i=1}^{n} x_i^{a-1}$, then

$$p(x,\theta) = h(\underline{x})\exp\{\eta T(\underline{x}) + B(\theta)\}.$$

So, X forms a one-parameter exponential family.

(c)

Since

$$p(x,\theta) = \theta^n a^{n\theta} / \prod_{i=1}^{n} x_i^{(\theta+1)} = \exp\{-(\theta+1)\sum_{i=1}^{n} \ln x_i + n\ln\theta + n\theta\ln a\},$$

Let $\eta = -(\theta+1)$, $T(\underline{x}) = \sum_{i=1}^{n} \ln x_i$, $B(\theta) = -n\ln\theta - n\theta\ln a$, and $h(\underline{x}) = 1$, then

$$p(x,\theta) = h(\underline{x})\exp\{\eta T(\underline{x}) + B(\theta)\},$$

So, X forms a one-parameter exponential family.

Problem 1.6.5
Show that the following families of distributions are two-parameter exponential families and identify the functions *η, B, T*, and *h*.
(a) The beta family.
(b) The gamma family.

Solution:
For Beta family,

$$p(x,\theta)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}I(0<x<1)$$
$$=\exp\{(\alpha-1)\ln x+(\beta-1)\ln(1-x)-\ln B(\alpha,\beta)\}I(0<x<1).$$

Let $\eta=(\alpha-1,\beta-1)'$, $T(x)=(\ln x,\ln(1-x))'$, $B(\eta)=\ln B(\alpha,\beta)$, and $h(x)=I(0<x<1)$.
Considering that if $\exists c_0,c_1,c_2 s.t. c_1\ln x+c_2\ln(1-x)=c_0$ is true for all $x\in(0,1)$, then there must exist $c_0=c_1=c_2=0$, hence $\ln x,\ln(1-x)$ are linearly independent. So, Beta family is two-parameter exponential family.
(b)
For Gamma family,

$$p(x,\theta)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-\frac{1}{\beta}}I(x>0)$$
$$=\exp\{(\alpha-1)\ln x-\frac{1}{\beta}\ln x-\ln\Gamma(\alpha)\beta^{\alpha}\}I(x>0).$$

Let $\eta=(\alpha-1,-\frac{1}{\beta})'$, $T(x)=(\ln x,x)'$, $B(\eta)=\ln\Gamma(\alpha)\beta^{\alpha}$, and $h(x)=I(x>0)$.
Considering that if $\exists c_0,c_1,c_2 s.t. c_1\ln x+c_2x=c_0$ is true for all x, then there must exist $c_0=c_1=c_2=0$, hence $\ln x,x$ are linearly independent. So, Gamma family is two-parameter exponential family.

Problem 1.6.6
Let *X* have the *Dirichlet* Distribution. $\mathcal{D}$(α), of Problem *1.2.15*. Show the distributions of *X* form a n r-parameter exponential family and identify *η, B, T*, and *h*.

Solution:
For *Dirichlet* family,

$$p(x,\theta)=\exp\{\sum_{i=1}^{r}(\alpha_i-1)\ln x_i+\ln\frac{\Gamma(\sum\alpha_i)}{\prod\alpha_i}\}I(0<x_i<1,\sum x_i=1).$$

Let $\eta=(\alpha_1-1,\alpha_2-1,...,\alpha_r-1)'$, $T(\underline{x})=(\ln x_1,...,\ln x_r)'$, $B(\eta)=-\ln\frac{\Gamma(\sum\alpha_i)}{\prod\alpha_i}$, and $h(x)=I(0<x_i<1,\sum x_i=1)$.

Considering that if $\exists c_0, c_1, ..., c_r s.t. \sum c_i \ln x_i = c_0$ is true for all x, then there must exist $c_0 = c_1 = \cdots = c_r = 0$, hence $\ln x_i$ are linearly independent. So *Dirichlet* family is r-parameter exponential family.

Problem 1.6.34

Let $(X_1, X_2, ..., X_n)$ be a stationary *Markov* chain with two states 0 and 1. That is,

$$P\left[X_i = \varepsilon_i \mid X_1 = \varepsilon_1, \ldots, X_{i-1} = \varepsilon_{i-1}\right] = P\left[X_i = \varepsilon_i \mid X_{i-1} = \varepsilon_{i-1}\right] = p_{\varepsilon_{i-1}, \varepsilon_i}$$

, where $\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ is the matrix of transition probabilities. Suppose further that

(i) $p_{00}=p_{11}=p$, so that $p_{10}=p_{01}=1-p$.

(ii) $P\left[X_1 = 0\right] = P\left[X_1 = 1\right] = \frac{1}{2}$

(a) Show that if $0<p<1$ is unknown this is a full rank, one-parameter exponential family with $T=N_{00}+N_{11}$ where $N_{ij} \equiv$ *the number of transitions from i to j*. For example, 01011 has $N_{01}=2, N_{11}=1, N_{00}=0,$ and $N_{10}=1$.

(b) Show that $E(T)=(n-1)p$ (by the method of indicators or otherwise).

<u>Solution:</u>

(a) Introduce indicator series $I_k, k = 1, ..., n-1$,

$$I_k = \begin{cases} 1 & \text{if } \mathrm{x_{k+1}} \text{ has the same status with } \mathrm{x_k} \\ 0 & \textit{otherwise} \end{cases}$$

Then by (i), we see that

$$P(I_k = 1) = p$$

, and

$$P(I_k = 0) = 1 - p.$$

Let $T = N_{00} + N_{11}$, $N_{ij} \equiv$ *numbers of transition from i to j.*

Then $T = N_{00} + N_{11} = \sum_k I_k \sim Bin(n-1, p)$;

And by the property of *Markov* Chain,

$$\begin{aligned} P(X_n, X_{n-1}, \ldots, X_1) &= P(X_n \mid X_{n-1}, X_{n-2}, \ldots, X_1) P(X_{n-1}, X_{n-2}, \ldots, X_1) \\ &= P(X_n \mid X_{n-1}) P(X_{n-1}, X_{n-2}, \ldots, X_1) \\ &= P(X_n \mid X_{n-1}) P(X_{n-1} \mid X_{n-2}) \cdots P(X_2 \mid X_1) P(X_1), \end{aligned}$$

, where $X_i = 0, or 1$ $i = 1, ..., n$ .As we know that $P_{00} = P_{11} = p$, $P_{10} = P_{01} = 1 - p$,

$$P(X_n, X_{n-1}, \ldots, X_1) = p^{N_{00}+N_{11}} (1-p)^{n-1-N_{00}-N_{11}} P(X_1),$$

, where $P(X_1 = 0) = P(X_1 = 1) = 1/2$ is given.

So for $\underline{x} = (x_1, ..., x_n)'$,

$$f(\underline{x}\,|\,p)=\{\frac{1}{2}p^{T}(1-p)^{n-1-T}+\frac{1}{2}p^{T}(1-p)^{n-1-T}\}\prod_i I(x_i\in\{0,1\})$$
$$=p^{T}(1-p)^{n-1-T}\prod_i I(x_i\in\{0,1\})$$
$$=\exp\{T\ln(\frac{p}{1-p})+(n-1)\ln(1-p)\}\prod_i I(x_i\in\{0,1\})$$

Let $\eta=\ln(\frac{p}{1-p})$, $T=T$, $B=-(n-1)\ln(1-p)$, and $h(\underline{x})=\prod_i I(x_i\in\{0,1\})$.

So $\underline{X}$ is a full rank, one-parameter exponential family.

(b)

$$E(T)=E(\sum\nolimits_k I_k)=(n-1)E(I_k)=(n-1)p\,.$$

2. For the set-up of problem 1.6.34, drop assumption (i) so that this becomes a 2 parameter problem. Identify a minimal sufficient statistic here (argue that it is minimal sufficient and describe it in simple terms).

Solution:

Drop assumption (i), then

$$f(\underline{x}\,|\,p_{00},p_{11})=(\frac{1}{2}+\frac{1}{2})p_{00}{}^{N_{00}}p_{11}{}^{N_{11}}(1-p_{00})^{N_{01}}(1-p_{11})^{n-1-N_{00}-N_{11}-N_{01}}$$
$$=p_{00}{}^{N_{00}}p_{11}{}^{N_{11}}(1-p_{00})^{N_{01}}(1-p_{11})^{n-1-N_{00}-N_{11}-N_{01}}$$

Now suppose that $\exists k(x,y)>0$ $s.t.$

$$f(y\,|\,p_{00},p_{11})=k(x,y)f(x\,|\,p_{00},p_{11})$$

i.e.

$$p_{00}{}^{N_{00}(x)}p_{11}{}^{N_{11}(x)}(1-p_{00})^{N_{01}(x)}(1-p_{11})^{n-1-N_{00}(x)-N_{11}(x)-N_{01}(x)}$$
$$=k(x,y)p_{00}{}^{N_{00}(y)}p_{11}{}^{N_{11}(y)}(1-p_{00})^{N_{01}(y)}(1-p_{11})^{n-1-N_{00}(y)-N_{11}(y)-N_{01}(y)}$$

holds for $\forall p_{00},p_{11}\in(0,1)$.

This equality requires that $N_{00}(x)=N_{00}(y),N_{11}(x)=N_{11}(y),N_{01}(x)=N_{01}(y)$.

(Otherwise, it is easy to give a counter-example.)

Let $T(\underline{X})=(N_{00}(\underline{X}),N_{11}(\underline{X}),N_{01}(\underline{X}))$, then $T(\underline{X})$ is minimal sufficient.

3. (Optional)

Suppose that $\Theta^*\subset\Theta$. Prove or give a counter-example for each of the following (where you need a counter-example, a simple discrete one with finite parameter space like the example Vardeman used to illustrate the Dynkin-Lehmann-Scheffe Theorem will be fine):

(a) $T(X)$ is sufficient for $\theta\in\Theta\Rightarrow T(X)$ is sufficient for $\theta\in\Theta^*$

(b) $T(X)$ is sufficient for $\theta\in\Theta^*\Rightarrow T(X)$ is sufficient for $\theta\in\Theta$

(c) Suppose that $T(X)$ is sufficient for $\theta \in \Theta$

i) $T(X)$ is minimal sufficient for $\theta \in \Theta \Rightarrow T(X)$ is minimal sufficient for $\theta \in \Theta^*$

ii) $T(X)$ is minimal sufficient for $\theta \in \Theta^* \Rightarrow T(X)$ is minimal sufficient for $\theta \in \Theta$.

Solution:

(a) *T(X)* is sufficient for $\theta \in \Theta$ means $X|T = t$ does not depend on all $\theta \in \Theta$, obliviously, does not depend those $\theta \in \Theta^*$, So *T(x)* is sufficient for $\theta \in \Theta^*$.

(b) Not holds, look at an example. Consider $X_1, \ldots, X_n$, *i.i.d*, $N(\mu,\sigma^2)$, then the natural parameter space is

$$\Theta = \left\{\left(\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right) : \mu \in R, \sigma^2 > 0\right\}$$

Then, the sufficient and minimal sufficient statistic is $T = \left(\sum x_i^2, \sum x_i\right)$. Now we are interested in $X_1, \cdots, X_n$, *i.i.d*, $N(\mu, 1)$, so the interested parameter space is

$$\Theta^* = \{(\mu, -1) : \mu \in R\},$$

And the sufficient and minimal sufficient statistic are $T = \sum x_i$. So T is sufficient for $\theta \in \Theta^*$ does not imply T is sufficient $\theta \in \Theta$.

(c)

i) Not holds, consider the example in (b), $T = \left(\sum x_i^2, \sum x_i\right)$ is minimal sufficient for $\theta \in \Theta$ does not imply T is sufficient for $\theta \in \Theta^*$.

ii) Not holds, consider the example in (b), $T = \sum x_i$ is minimal sufficient for $\theta \in \Theta^*$ does not imply T is sufficient $\theta \in \Theta$.

4.

Problem 1.5.12 (*Optional*)

Dynkin.Lehmann, Scheffés Theorem. Let $P = \{P_\theta : \theta \in \Theta\}$ where P_θ is discrete concentrated on $\mathcal{X}=\{x_1, x_2, \ldots\}$. Let $p(x,\theta) \equiv P_\theta[X = x] \equiv L_x(\theta) > 0$ on $\mathcal{X}$. Show that $\frac{L_X(.)}{L_X(\theta_0)}$ is minimal sufficient.

Hint: Apply the factorization theorem.

Solution:

Since $\forall\ x \in \mathcal{X},\ \theta \in \Theta, L_x(\theta) > 0$, so $R(X,.) = \frac{L_X(.)}{L_X(\theta_0)}$ is well defined.

$$L(X,\theta) = \frac{L_X(.)}{L_X(\theta_0)} L_X(\theta_0), h(x) = L_X(\theta_0)$$

, then by factorization theorem, $R(X,.)$ is sufficient.

For any other sufficient statistic *T*, by factorization theorem, $L(X,\theta) = g(T,\theta)h_1(x)$, so

$$R(X,.) = \frac{g(T,.)h_1(x)}{g(T,\theta_0)h_1(x)} = \frac{g(T,.)}{g(T,\theta_0)},$$

So $R(X,.)$ is a function of *T*, and thus $R(X,.) = \dfrac{L_X(.)}{L_X(\theta_0)}$ is minimal sufficient.

Problem 1.6.15 (Optional)

Let $P = \{P_\theta : \theta \in \Theta\}$ where P_θ is discrete and concentrated on $\mathcal{X}=\{x_1, x_2, \ldots\}$. And let $p(x,\theta) = P_\theta[X = x]$. Show that if $\mathcal{P}$ is a (discrete) canonical exponential family generated by (***T***,*h*) and $\varepsilon^0 \neq \varnothing$, then **T** is minimal sufficient.

Hint: $\dfrac{\partial \log}{\partial \eta_j} L_X(\eta) = T_j(X) - E_\eta T_j(X)$. Use Problem 1.5.12.

<u>Solution:</u>

The exponential family generated by *(T, h)* is $h(x)e^{T'\eta - A(\eta)}$, by problem 1.5.12,

$$R(X,\eta) = \frac{e^{T'\eta - A(\eta)}}{e^{T'\eta_0 - A(\eta_0)}},$$

is minimal sufficient, and *R(X,.)* is a function of *T*.

Since $\sum^0 \neq \varnothing$, by corollary 1.6.1,

$$\frac{\partial Log L_X(\eta)}{\partial \eta} = T(X) - E_\eta T(X)$$

, and so

$$\frac{\partial Log R_X(X,\eta)}{\partial \eta} = T(X) - E_\eta T(X)$$

, then $T = \dfrac{\partial Log R_X(X,\eta)}{\partial \eta} + E_\eta T(X)$, which means that *T* is a function of *R(X,.)*.

So, *T* and *R(X,.)*, are equivalent, and hence *T* is minimal sufficient.

1.6.22 (Optional)

Let $X_1, X_2, \ldots, X_n$ be a sample from the k-parameter exponential family distribution (1.6.10). Let $T = \left(\sum_{i=1}^n T_1(X_i), \ldots, \sum_{i=1}^n T_k(X_i)\right)$ and let

$$S = \{(\eta_1(\theta), \ldots, \eta_k(\theta)) :, \theta \in \Theta\}$$

Show that if *S* contains a subset of k+1 vectors $v_0, \ldots, v_{k+1}$ so that $v_i - v_0, 1 \le i \le k,$ are not collinear (linearly independent), then *T* is minimally sufficient for θ.

Solution I:

As $X_1, \ldots X_n$ be sample from k-parameters exponential family ,we have

$$p\left(X \mid \theta\right) = h(X)\exp\left\{T'\eta - B(\theta)\right\}$$

Suppose S is a sufficient statistic for the sample exponential family. Then by factori zation theorem,

$$p(X \mid \theta) = g(S \mid \theta)\tilde{h}(X).$$

Thus,

$$\frac{p(X \mid \theta)}{p(X \mid \theta_0)} = \frac{g(S \mid \theta)}{g(S \mid \theta_0)} = \frac{\exp(T'\eta - B(\theta))}{\exp(T'\eta_0 - B(\theta_0))} = \exp\left(T'(\eta - \eta_0) - B(\theta) + B(\theta_0)\right),$$

by taking *log* of the above equation, $T'\left(\eta - \eta_0\right) - B(\theta) + B(\theta_0) = \log\left\{g(S,\theta)/g(S,\theta_0)\right\}$. If S contains a subset of $k+1$ vectors $v_0, \ldots, v_{k+1}$, such that $v_i - v_0$, $1 \le i \le k$ are linearly independent, which means there exist $u_i = \eta_i - \eta_0$, $1 \le i \le k$ are linearly independent.. Then we can solve $\boldsymbol{T} = (T_1, T_2, \ldots, T_k)$ by S from the linear equation established. Thus $\boldsymbol{T}$ is minimal sufficient for θ.

Solution II:

What we need to do is to show there is an open rectangle is S, if such rectangle exists, Then by theorem, the natural sufficient statistic is minimal sufficient. Since $v_0, v_1, \ldots, v_k \in S$ and S is convex, then the convex combination of $v_0, v_1, \ldots, v_k$ should still in S, let $A_0 = \frac{1}{k+1}\sum_{i=0}^{k} v_i$, $v_i \in S$.

We will build an rectangle A with one acme is A_0 and with length $\varepsilon > 0$ but ε is small enough. The acme of A is $A_i = (a_1, \ldots, a_k)$, $a_j = 0$ or ε $j = 1, \ldots, k$. and $A_0 = \sum_{j=0}^{k} \lambda_j v_j$.

We will show that such $\lambda > 0$ exists for every A_i. So there is an open rectangle in S, so T is minimal sufficient.

To show λ exists, we need to solve the linear function system,

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ v_{01} & v_{11} & \ldots & v_{k1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{0k} & v_{1k} & \cdots & v_{kk} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{k+1}\sum v_{j1} \\ \vdots \\ \frac{1}{k+1}\sum v_{jk} \end{pmatrix} + \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_k \end{pmatrix}$$

Since $v_1 - v_0, \ldots, v_k - v_0$ are linear independent ,so,

$$rank\left(\begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_0 & v_1 & \ldots & v_k \end{pmatrix}\right) = rank\left(\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & v_1 - v_0 & \ldots & v_k - v_0 \end{pmatrix}\right) = k+1$$

, so $M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & v_1 - v_0 & \ldots & v_k - v_0 \end{pmatrix}$ is invariable, so

$$\lambda = M^{-1} \begin{pmatrix} 1 \\ \frac{1}{k+1} \sum v_{j1} \\ \vdots \\ \frac{1}{k+1} \sum v_{jk} \end{pmatrix} + M^{-1} \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_k \end{pmatrix} = \frac{1}{k+1} \begin{pmatrix} 1, & \cdots, & 1 \end{pmatrix}' + M^{-1} \begin{pmatrix} 0 & a_1 & \cdots & a_k \end{pmatrix}'$$

Since ε is small enough, so $\lambda > 0$ exists for all A_i.