Homework 9 Solution

1.

(Problem 4.9.1)

Let X have a binomial, $B(n,\theta)$, distritution. Show that the likelihood ratio statistic for testing $H:\theta=\frac{1}{2}$ versus $K:\theta\neq\frac{1}{2}$ is equivalent to $|2X-n|$.

Hint: Show that for $x\leq\frac{1}{2}n$, $\lambda(x)$ is an increasing funciton of $-(2x-n)$ and $\lambda(x)=\lambda(n-x)$.

Solution:

From $X\tilde{}\,Binomial(n,\theta)$,

$\Rightarrow L(\theta)=f_X(x)=\binom{n}{x}\theta^x(1-\theta)^{n-x}$

$\Rightarrow \log L(\theta)=\log\binom{n}{x}+x\log\theta+(n-x)\log(1-\theta)$

By setting $\frac{\partial\log L(\theta)}{\partial\theta}=\frac{x}{\theta}-\frac{n-x}{1-\theta}=0,\Rightarrow\hat{\theta}_{MLE}=\frac{x}{n}$.

Then, $\lambda(x)=\frac{\underset{\theta\in\Theta}{Sup}L(\theta)}{\underset{\theta\in\Theta_0}{Sup}\,L(\theta)}=\frac{L(\hat{\theta}_{MLE})}{L(\frac{1}{2})}=\frac{\binom{n}{x}\left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x}}{\binom{n}{x}\left(\frac{1}{2}\right)^n}=\left(\frac{2}{n}\right)^n x^x(n-x)^{n-x}$

$\Rightarrow\log\lambda(x)=n\log\left(\frac{2}{n}\right)+x\log x+(n-x)\log(n-x)$.

Let $t\hat{=}x-\frac{n}{2}$, then

$\log\lambda(t)=n\log\left(\frac{2}{n}\right)+\left(t+\frac{n}{2}\right)\log\left(t+\frac{n}{2}\right)+\left(\frac{n}{2}-t\right)\log\left(\frac{n}{2}-t\right)$

$$\begin{aligned}\Rightarrow\frac{\partial\log\lambda(t)}{\partial t}&=\log\left(t+\frac{n}{2}\right)+1-\log\left(\frac{n}{2}-t\right)-1\\&=\log\left(t+\frac{n}{2}\right)-\log\left(\frac{n}{2}-t\right)\end{aligned}$$

From $\begin{cases}t+\frac{n}{2}>0\\\frac{n}{2}-t>0\end{cases}\Rightarrow-\frac{n}{2}<t<\frac{n}{2}$

When $t>0,\Rightarrow t+\frac{n}{2}>\frac{n}{2}-t\Rightarrow\frac{\partial\log\lambda(t)}{\partial t}>0$

$\Rightarrow\log\lambda(t)$ is an increasing function of t ; i.e. $\lambda(t)$ is an increasing function of $|t|$;

When $t<0,\Rightarrow t+\frac{n}{2}<\frac{n}{2}-t\Rightarrow\frac{\partial\log\lambda(t)}{\partial t}<0$

$\Rightarrow\log\lambda(t)$ is an decreasing function of t, i.e. $\lambda(t)$ is an decreasing function of $|t|$;

Hence, $\lambda(t)$ is an increasing function of $|t|$, i.e. $\lambda(x)$ is an increasing function of $|2x-n|$, and then the likelihood ratio statistic for testing $H:\theta=\frac{1}{2}$ versus $K:\theta\neq\frac{1}{2}$ is equivalent to $|2X-n|$.

(Problem 4.9.3)

Let $X_1,...,X_n$ be a $N(\mu,\sigma^2)$ sample with both μ and σ^2 unknown.

One-Sied Tests for Scale. We want to test $H:\sigma^2\leq\sigma_0^2$ versus $K:\sigma^2>\sigma_0^2$. Show that

(a) Likelihood ratio tests are of the form: Reject if, and only if,

$$\frac{n\hat{\sigma}^2}{\sigma_0^2}=\frac{1}{\sigma_0^2}\sum_{i=1}^n\left(X_i-\bar{X}\right)^2\geq c.$$

Hint: $\log\lambda(x)=0$, if $\frac{\hat{\sigma}^2}{\sigma_0^2}\leq1$ and $\log\lambda(x)=\frac{n}{2}\left[\frac{\hat{\sigma}^2}{\sigma_0^2}-1-\log\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)\right]$ otherwise.

(b) To obtain size α for H we should take $c=x_{n-1}(1-\alpha)$.

Hint: Recall Theorem B.3.3.

(c) These tests coincide with the tests obtained by inverting the family of level $(1-\alpha)$ lower confidence bounds for σ^2.

Solution:

(a) For $X_1, ..., X_n \tilde{} N(\mu, \sigma^2)$, both μ and σ^2 unknown,

$$L(\mu, \sigma^2) = f(x_1, ..., x_n) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum (x_i - \mu)^2\right\}$$

$\Rightarrow \hat{\mu}_{MLE} = \bar{x}, \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum (x_i - \bar{x})^2$

Then, $\lambda(x_1, ..., x_n) = \frac{\underset{\theta\in\Theta}{Sup}\ L(\mu,\sigma^2)}{\underset{\theta\in\Theta_0}{Sup}\ L(\mu,\sigma^2)}$

$$= \begin{cases} \dfrac{\left(2\pi\hat{\sigma}^2_{MLE}\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}^2_{MLE}}\sum(x_i - \hat{\mu}_{MLE})^2\right\}}{\left(2\pi\hat{\sigma}^2_{MLE}\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}^2_{MLE}}\sum(x_i - \hat{\mu}_{MLE})^2\right\}} = 1 & if\ \hat{\sigma}^2_{MLE}/\sigma_0^2 \le 1 \\ \dfrac{\left(2\pi\hat{\sigma}^2_{MLE}\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}^2_{MLE}}\sum(x_i - \hat{\mu}_{MLE})^2\right\}}{\left(2\pi\hat{\sigma}^2_0\right)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}^2_0}\sum(x_i - \hat{\mu}_{MLE})^2\right\}} & if\ \hat{\sigma}^2_{MLE}/\sigma_0^2 > 1 \end{cases}$$

$$\Rightarrow \log\lambda(x_1, ..., x_n) = \begin{cases} 1 & if\ \hat{\sigma}^2_{MLE}/\sigma_0^2 \le 1 \\ \frac{n}{2}\left[-\log\frac{\hat{\sigma}^2_{MLE}}{\sigma_0^2} - 1 + \frac{\hat{\sigma}^2_{MLE}}{\sigma_0^2}\right] & if\ \hat{\sigma}^2_{MLE}/\sigma_0^2 > 1 \end{cases}$$

Since $\log\lambda(x_1, ..., x_n)$ is an increasing function of $\frac{\hat{\sigma}^2_{MLE}}{\sigma_0^2}$, i.e. $\lambda(x_1, ..., x_n)$ is an increasing function of $\frac{n\hat{\sigma}^2_{MLE}}{\sigma_0^2}$, then the Likelihood ratio tests can be of the form:

Reject if, and only if, $\frac{n\hat{\sigma}^2}{\sigma_0^2} = \frac{1}{\sigma_0^2}\sum_{i=1}^n \left(X_i - \bar{X}\right)^2 \ge c.$

(b) By theorem B.3.3, under the null hypothesis, $\frac{n\hat{\sigma}^2}{\sigma_0^2} = \frac{1}{\sigma_0^2}\sum_{i=1}^n \left(X_i - \bar{X}\right)^2$ is in χ^2_{n-1} distribution. To obtain size α for H, $P\left(\frac{1}{\sigma_0^2}\sum_{i=1}^n \left(X_i - \bar{X}\right)^2 \ge c\right) = \alpha \Rightarrow c = \chi^2_{n-1}(1-\alpha)$.

(c) By the MLR, this test is UMP test. And by duality theorem, this test form a $1-\alpha$ confidence set for σ^2. Still by MLR, We have that this test is equivalent to inverting the $1-\alpha$ lower confidence bound.

(Problem 4.9.9)

The normally distributed random variables $X_1, ..., X_n$ are said to be *serially correlated* or to follow an autoregressive model if we can write

$$X_i = \theta X_{i-1} + \varepsilon_i,\ i = 1, ..., n,$$

where $X_0 = 0$ and $\varepsilon_1, ..., \varepsilon_n$ are independent $N(0, \sigma^2)$ random variables.

(a) Show that the density of $X = (X_1, ..., X_n)$ is

$$p(x, \theta) = (2\pi\sigma^2)^{-\frac{1}{2}n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta x_{i-1})^2\right\}$$

for $-\infty < x_i < \infty, i = 1, ..., n, x_0 = 0$.

(b) Show that the likelihood ratio statistic of $H : \theta = 0$ (independence) versus $K : \theta \ne 0$ (serial correlation) is equivalent to $-\left(\sum_{i=2}^n X_i X_{i-1}\right)^2 / \sum_{i=1}^{n-1} X_i^2$.

Solution:

(a) From $X_i = \theta X_{i-1} + \varepsilon_i,\ i = 1, ..., n,$ where $X_0 = 0$ and $\varepsilon_1, ..., \varepsilon_n i.\underset{\sim}{i}.d.\ N(0, \sigma^2)$,

$\Rightarrow X_i | X_{i-1}, ..., X_0 \tilde{} N(\theta X_{i-1}, \sigma^2)$

i.e. $P(x_i | x_{i-1}, ..., x_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \theta x_{i-1})^2\right\}$

Then, $P(x_1, ..., x_n)$

$= P(x_n | x_{n-1}, ..., x_0) P(x_{n-1} | x_{n-2}, ..., x_0) \cdots P(x_2 | x_1) P(x_1)$

$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta x_{i-1})^2\right\}$

(b) $\log L(\theta) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta x_{i-1})^2$

$\frac{dP(x_1,\ldots,x_n|\theta)}{d\theta} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_{i-1}\left(x_i - \theta x_{i-1}\right) = 0$
$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum_{i=2}^{n} x_{i-1}x_i}{\sum_{i=1}^{n-1} x_i^2}$
The Likelihood Ratio is

$$\lambda(x) = \frac{\underset{\theta\in\Theta}{Sup}\ L(\theta)}{\underset{\theta\in\Theta_0}{Sup}\ L(\theta)} = \frac{\left(2\pi\sigma^2\right)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(x_i-\hat{\theta}_{MLE}x_{i-1}\right)^2\right\}}{\left(2\pi\sigma^2\right)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2\right\}}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\hat{\theta}_{MLE}^2 x_{i-1}^2 - 2\hat{\theta}_{MLE}x_{i-1}x_i\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\left(\sum_{i=2}^{n}x_{i-1}x_i\right)^2}{\left(\sum_{i=1}^{n-1}x_i^2\right)^2}\left(\sum_{i=1}^{n-1}x_i^2\right) - 2\frac{\left(\sum_{i=2}^{n}x_{i-1}x_i\right)^2}{\sum_{i=1}^{n-1}x_i^2}\right]\right\}$$

$$= \exp\left\{\frac{1}{2\sigma^2}\frac{\left(\sum_{i=2}^{n}x_{i-1}x_i\right)^2}{\sum_{i=1}^{n-1}x_i^2}\right\}$$

$\Rightarrow \lambda(x)$ is an increasing function of $\frac{\left(\sum_{i=2}^{n}x_{i-1}x_i\right)^2}{\sum_{i=1}^{n-1}x_i^2}$.
So, the likelihood ratio statistic of $H : \theta = 0$ (independence) versus $K : \theta \neq 0$ (serial correlation) is equivalent to $\frac{\left(\sum_{i=2}^{n}x_{i-1}x_i\right)^2}{\sum_{i=1}^{n-1}x_i^2}$.

(Problem 4.9.10)
(An example due to C.Stein). Consider the following model. Fix $0 < \alpha < \frac{1}{2}$ and $\frac{\alpha}{[2(1-\alpha)]} < c < \alpha$. Let Θ consist of the point -1 and the interval $[0,1]$. Define the frequency functions $p(x,\theta)$ by the following table.

$\theta \backslash x$	-2	-1	0	1	2
-1	$\frac{1}{2}\alpha$	$\frac{1}{2}-\alpha$	α	$\frac{1}{2}-\alpha$	$\frac{1}{2}\alpha$
$\neq -1$	θc	$\left(\frac{1-c}{1-\alpha}\right)\left(\frac{1}{2}-\alpha\right)$	$\left(\frac{1-c}{1-\alpha}\right)\alpha$	$\left(\frac{1-c}{1-\alpha}\right)\left(\frac{1}{2}-\alpha\right)$	$(1-\theta)c$

(a) What is the size α likelihood ratio test for testing $H : \theta = -1$ versus $K : \theta \neq -1$?
(b) Show that the test that rejects if, and only if, $X = 0$, has level α and is strictly more powerful whatever be θ.
Solution:
(a) $\frac{\alpha}{[2(1-\alpha)]} < c < \alpha \Rightarrow -\alpha < -c\ \&\frac{1}{1-\alpha} < \frac{2c}{\alpha} \Rightarrow 1 < \frac{1-c}{1-\alpha} < \frac{1}{1-\alpha} < \frac{2c}{\alpha}$
The likelihood ratio is $\frac{\underset{\theta\in\Theta}{Sup}\ L(\theta)}{\underset{\theta\in\Theta_0}{Sup}\ L(\theta)}$, which can be listed below as corresponding different x observed.

x	-2	-1	0	1	2
LR	$\frac{2\theta c}{\alpha}$	$\frac{1-c}{1-\alpha}$	$\frac{1-c}{1-\alpha}$	$\frac{1-c}{1-\alpha}$	$\frac{2(1-\theta)c}{\alpha}$

From $\alpha = P_{\theta_0}(LR \geq c) = P(LR \geq c|\theta_0 = -1) = P(x = \pm 2)$,
$\Rightarrow$ The size α likelihood ratio test for testing $H : \theta = -1$ versus $K : \theta \neq -1$ is

$$\phi(x) = \begin{array}{ll} 1 & if\ x = \pm 2 \\ 0 & otherwise \end{array}$$

(b) The size α test suggested here is in the form of

$$\phi(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & otherwise \end{cases}$$

and $E_{\theta_1}(\phi(x)) = \frac{1-c}{1-\alpha}\alpha$, for $\theta_1 \in \Theta_1$.
For the test in (a), $E_{\theta_1}(\phi(x)) = c$, for $\theta_1 \in \Theta_1$.

Since $\frac{1-c}{1-\alpha}\alpha - c = \frac{\alpha - c\alpha - c + c\alpha}{1-\alpha} = \frac{\alpha - c}{1-\alpha} > 0$, thus the test here is strictly more powerful whatever be θ.

2

(Problem 4.7.2)

Suppose that given $\lambda = \lambda, X_1, ..., X_n$ are i.i.d Proisson, $\wp(\lambda)$ and that λ is distributed as $\frac{V}{s_0}$, where s_0 is some constant and $V\tilde{}\chi_k^2$. Let $T = \sum_{i=1}^n X_i$.

(a) Show that $(\lambda|T = t)$ is distributed as $\frac{W}{s}$, where $s = s_0 + 2n$ and $W\tilde{}\chi_m^2$ with $m = k + 2t$.

(b) Show how quantiles of the χ^2 distribution can be used to determine level $(1-\alpha)$ upper and lower credible bounds for λ.

Solution:

(a) $T\tilde{}\sum_{i=1}^n X_i\tilde{}Poisson\left(n\lambda\right), \Rightarrow P\left(T = t|\lambda\right) = \frac{e^{-n\lambda}(n\lambda)^t}{t!}$

and $f\left(\lambda\right) \propto e^{-\frac{s_0\lambda}{2}}\left(s_0\lambda\right)^{\frac{k}{2}-1} \Rightarrow f\left(T = t, \lambda\right) \propto P\left(T = t|\lambda\right) f\left(\lambda\right) \propto e^{-\frac{\lambda(s_0+2n)}{2}}\lambda^{\frac{k+2t}{2}-1}$

Let $W = \lambda\left(s_0 + 2n\right)$, i.e. $\lambda = \frac{W}{s_0+2n}$, then

$f\left(w\right) \propto e^{-\frac{w}{2}}w^{\frac{k+2t}{2}-1}\tilde{}\chi_{k+2t}^2$

$\Rightarrow \lambda|T$ is distributed as $\frac{W}{s}$, $s = s_0 + 2n$, and $W\tilde{}\chi_m^2$, $m = k + 2t$.

(b) $P\left(\chi^2_{m,\frac{\alpha}{2}} \le W \le \chi^2_{m,1-\frac{\alpha}{2}}\right) = 1 - \alpha$

$\Rightarrow P\left(\frac{\chi^2_{m,\frac{\alpha}{2}}}{s} \le \frac{W}{s} \le \frac{\chi^2_{m,1-\frac{\alpha}{2}}}{s}\right) = 1 - \alpha$

$\Rightarrow \left(\frac{\chi^2_{m,\frac{\alpha}{2}}}{s}, \frac{\chi^2_{m,1-\frac{\alpha}{2}}}{s}\right)$ is the $(1-\alpha)$ lower and upper credible bounds for λ.

(Problem 4.4.1)

Let $X_1, ..., \cancel{X}_n$ be a sample from a normal population with unknown mean μ and unknown variance σ^2. Use a pivot based on $\sum_{i=1}^n \left(X_i - \bar{X}\right)^2$,

(a) Show how to construct level $(1-\alpha)$ confidence intervals of fixed finite length for $\log\sigma^2$.

(b) Suppose that $\sum_{i=1}^n (X_i - \bar{X})^2 = 16.52, n = 2, \alpha = 0.01$. What would you announce as your level $(1-\alpha)$ UCB for σ^2?

Solution:

(a) We've known that $S^2 = \frac{1}{n-1}\sum_i \left(x_i - \bar{x}\right)^2$, and $\frac{(n-1)S^2}{\sigma^2}\tilde{}\chi^2_{n-1}$.

Then, the $(1-\alpha)$ confidence interval for σ^2 is $\left(\frac{(n-1)s^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}, \frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}\right)$,

and the $(1-\alpha)$ confidence interval for $\log\sigma^2$ is $\left(\log\left(\frac{(n-1)s^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right), \log\left(\frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}\right)\right)$.

(b) From $\frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\chi^2_{2-1}\left(\frac{0.01}{2}\right)} = \frac{16.52}{3.927042\times 10^{-5}} = 420672.8$, we see that level $(1-\alpha)$ UCB is too large.

Think about $\frac{(n-1)s^2}{\chi^2_{n-1}(\alpha)} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\chi^2_{2-1}(0.01)} = \frac{16.52}{0.0001570879} = 105164.1 < 420672.8$,

so $\left(0, \frac{(n-1)s^2}{\chi^2_{n-1}(\alpha)}\right)$ may be a better level $(1-\alpha)$ UCB for σ^2.

(Problem 4.4.2)
Let $X_i = \frac{\theta}{2}t_i^2 + \varepsilon_i, i = 1, ..., n$, where the ε_i are independent normal random variables with mean 0 and known variance σ^2 (cf. Problem 2.2.1).
(a) Using a pivot based on the MLE $\frac{2\sum_{i=1}^n t_i^2 X_i}{\sum_{i=1}^n t_i^4}$ of θ, find a fixed length level $(1-\alpha)$ confidence interval for θ.
(b) If $0 \le t_i \le 1, i = 1, ..., n$, but we may otherwise choose the t_i freely, what values should we use for the t_i so as to make our interval as short as possible for given α?
Solution:
(a) Since $\frac{2\sum_{i=1}^n t_i^2 X_i}{\sum_{i=1}^n t_i^4} - \theta = \frac{2\sum_{i=1}^n t_i^2\left(X_i - \frac{\theta}{2}t_i^2\right)}{\sum_{i=1}^n t_i^4} = \frac{2\sum_{i=1}^n t_i^2 \varepsilon_i}{\sum_{i=1}^n t_i^4} \sim N\left(0, \frac{4\sigma^2}{\sum_{i=1}^n t_i^4}\right)$,
then the fixed length level $(1-\alpha)$ confidence interval for θ is

$$\left(\frac{2\sum_{i=1}^n t_i^2 X_i}{\sum_{i=1}^n t_i^4} - Z\left(1-\frac{\alpha}{2}\right)\frac{2\sigma}{\sqrt{\sum_{i=1}^n t_i^4}}, \frac{2\sum_{i=1}^n t_i^2 X_i}{\sum_{i=1}^n t_i^4} + Z\left(1-\frac{\alpha}{2}\right)\frac{2\sigma}{\sqrt{\sum_{i=1}^n t_i^4}}\right).$$

(b) $\frac{4\sigma^2}{\sum_{i=1}^n t_i^4} \ge \frac{4\sigma^2}{n}$, the equality holds iff $t_1 = \cdots t_n = 1$
$\Rightarrow$ To make our interval as short as possible means to make variance as small as possible, so we should use $t_1 = \cdots t_n = 1$.

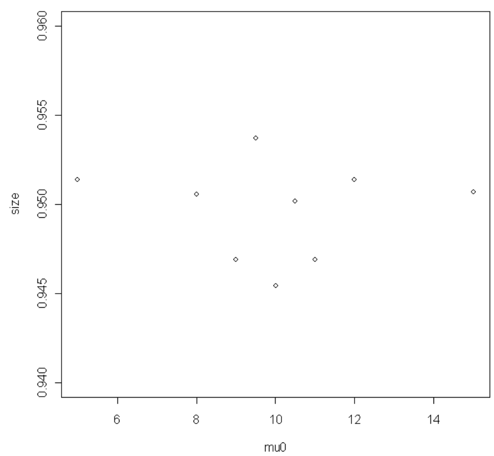
(Problem 4.8.1)
Let $X_1, ...X_{n+1}$ be i.i.d. as $X \tilde{} N(\mu, \sigma_0^2)$, where σ_0^2 is known. Here $X_1, ..., X_n$ is obsevable and X_{n+1} is to be predicted.
(a) Give a level $(1-\alpha)$ prediction interval for X_{n+1}.
(b) Compare the interval in part (a) to the Bayesian prediction interval (4.8.3) by doing a frequentist computation of the probability of coverage. That is, suppose $X_1, ..., X_n$ are i.i.d. $N(\mu, \sigma_0^2)$. Take $\sigma_0^2 = \tau^2 = 1, N = 100, \eta_0 = 10$, and $\alpha = 0.05$. Then the level of the frequentist interval is 95%. Find the probability that the Bayesian interval covers the true mean μ for $\mu = 5, 8, 9, 9.5, 10, 10.5, 11, 12, 15$. Present the results in a table and a graph.
Solution:
(a) Since $\frac{x_{n+1}-\bar{x}}{\sqrt{1+\frac{1}{n}}} \sim N(0, \sigma_0^2)$
$\Rightarrow$ level $(1-\alpha)$ prediction interval fok x_{n+1} is $\left(\bar{x} - Z_{1-\frac{\alpha}{2}}\sqrt{1+\frac{1}{n}}\sigma_0, \bar{x} + Z_{1-\frac{\alpha}{2}}\sqrt{1+\frac{1}{n}}\sigma_0\right)$.
(b)From Expample 4.8.3, level $(1-\alpha)$ Bayesian prediction interval for Y is $\left(Y_B^-, Y_B^+\right)$,
where $Y_B^{\pm} = \hat{\mu}_B \pm Z\left(1-\frac{1}{2}\alpha\right)\sqrt{\sigma_0^2 + \hat{\sigma}_B^2}$,
and $\hat{\sigma}_B^2 = \frac{1}{\frac{n}{\sigma_0^2}+\frac{1}{\tau^2}}$, $\hat{\mu}_B = \frac{\sigma_B^2}{\tau^2}\eta_0 + \frac{n\sigma_B^2}{\sigma_0^2}\bar{x}$.
Plug in $\sigma_0^2 = \tau^2 = 1, N = 100, \eta_0 = 10$, and $\alpha = 0.05$, we got level $(1-\alpha)$ Bayesian prediction interval for Y is

$$\left(\frac{10}{101} + \frac{100}{101}\bar{x} - 1.06\sqrt{\frac{102}{101}}, \frac{10}{101} + \frac{100}{101}\bar{x} + 1.06\sqrt{\frac{102}{101}}\right).$$

Simulate data from $N(\mu, \sigma_0^2)$, with $\mu = 5, 8, 9, 9.5, 10, 10.5, 11, 12, 15$, respectively.
The simulation results are shown in table as
mu cprob
1 5.0 0.9514
2 8.0 0.9506
3 9.0 0.9469
4 9.5 0.9537

5 10.0 0.9454
6 10.5 0.9502
7 11.0 0.9469
8 12.0 0.9514
9 15.0 0.9507

And they are displayed in graph as



(Problem 4.8.3)
Suppose $X_1, ..., X_{n+1}$ are i.i.d. as X where X has the exponential distribution

$$F(x|\theta) = 1 - e^{-\frac{x}{\theta}},\ x > 0, \theta > 0.$$

Suppose $X_1, ..., X_n$ are observable and we want to predict X_{n+1}. Give a level $(1-\alpha)$ prediction interval for X_{n+1}.
Hint: $\frac{X_i}{\theta}$ has a χ_2^2 distribution and $\frac{nX_{n+1}}{\sum_{i=1}^n X_i}$ has an $F_{2,2n}$ distribution.
Solution:
Since $X_i \sim \exp(\theta)$, $f(x_i) = \frac{1}{\theta} e^{-\frac{x_i}{\theta}}, \Rightarrow \frac{X_i}{\theta} \sim \chi_2^2$
Then, $\sum_{i=1}^n \frac{X_i}{\theta} \sim \chi_{2n}^2$, and thus $\frac{X_{n+1}}{2\theta} / \frac{\sum_{i=1}^n X_i}{2n\theta} \sim F_{2,2n}$, i.e. $\frac{nX_{n+1}}{\sum_{i=1}^n X_i} \sim F_{2,2n}$
So, a level $(1-\alpha)$ prediction interval for X_{n+1} is $\left(F_{2,2n}\left(\frac{\alpha}{2}\right) \frac{\sum_{i=1}^n X_i}{n}, F_{2,2n}\left(1 - \frac{\alpha}{2}\right) \frac{\sum_{i=1}^n X_i}{n} \right)$.

3.
(Problem 4.5.1)
Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be independent exponential $\epsilon(\theta)$ and $\epsilon(\lambda)$ samples, respectively, and let $\triangle = \frac{\theta}{\lambda}$.
(a) If $f(\alpha)$ denotes the αth quantile of the $F_{2n_1,2n_2}$ distribution, show that $\left[\bar{Y} f\left(\frac{1}{2}\alpha\right) / \bar{X}, \bar{Y} f\left(1 - \frac{1}{2}\alpha\right) / \bar{X}\right]$ is a confidence interval for $\triangle$ with confidence coefficient $1-\alpha$.
Hint: Use the results of Problems B.3.4 and B.3.5.

(b) Show that the test with acceptance region $\left[f\left(\frac{1}{2}\alpha\right) \leq \bar{X}/\bar{Y} \leq f\left(1-\frac{1}{2}\alpha\right)\right]$ has size α for testing $H : \triangle = 1$ versus $K : \triangle \neq 1$.

(c) The following are times until breakdown in days of air monitors operated under two different maintenance policies at a nuclear power plant. Experience has shown that the exponential assumption is warranted. Give a 90% confidence interval for the ratio $\triangle$ of mean life times.

x	3	150	40	34	32	37	34	2	31	6	5	14	150	27	4	6	27	10	30	37
y	8	26	10	29	20	10														

Is $H : \triangle = 1$ rejected at level $\alpha = 0.10$?

Solution:

(a) Since $X_1, ..., X_{n_1} \tilde{} E(\theta), Y_1, ..., Y_{n_2} \tilde{} E(\lambda)$, then $X_i\theta \tilde{} \chi^2_2, Y_i\lambda \tilde{} \chi^2_2$

$\Rightarrow \frac{\theta \sum_{i=1}^{n_1} X_i}{n_1} / \frac{\lambda \sum_{i=1}^{n_2} Y_i}{n_2} \tilde{} F_{2n_1,2n_2} \Rightarrow \frac{\theta}{\lambda}\frac{\bar{x}}{\bar{y}} \tilde{} F_{2n_1,2n_2}, \triangle = \frac{\theta}{\lambda}$

$\Rightarrow (1-\alpha)$ level confidence interval for $\triangle$ is $\left(\bar{Y}f\left(\frac{1}{2}\alpha\right)/\bar{X}, \bar{Y}f\left(1-\frac{1}{2}\alpha\right)/\bar{X}\right)$;

(b) $P\left(f\left(\frac{1}{2}\alpha\right) \leq \frac{\bar{X}}{\bar{Y}}\triangle \leq f\left(1-\frac{1}{2}\alpha\right)\right) = 1-\alpha$

$\Leftrightarrow P\left(\bar{Y}f\left(\frac{1}{2}\alpha\right)/\bar{X} \leq \triangle \leq \bar{Y}f\left(1-\frac{1}{2}\alpha\right)/\bar{X}\right) = 1-\alpha$

So, the test with acceptance region $\left[f\left(\frac{1}{2}\alpha\right) \leq \bar{X}/\bar{Y} \leq f\left(1-\frac{1}{2}\alpha\right)\right]$ has size α for testing $H : \triangle = 1$ versus $K : \triangle \neq 1$.

(c) $n_1 = 20, n_2 = 7, \bar{x} = 33.95, \bar{y} = 15.8571$

$F_{210,14,0.05} = 0.5134432, F_{40,14,0.95} = 2.266350$

$\Rightarrow$ 90% CI for $\triangle$ is $(0.2398157, 1.058552)$

$\Rightarrow \triangle = 1$ is not rejected at level 90%.

4. (Optional)
(Problems 4.4.7)

Suppose we want to select a sample size N such that the interval (4.4.1) based on $n = N$ observations has length at most l for some preassigned length $l = 2d$. Stein's (1945) two-stage procedure is the following. Begin by taking a fixed number $n_0 \geq 2$ of observations and calculate $\bar{X}_0 = \frac{1}{n_0}\sum_{i=1}^{n_0} X_i$ and

$$s_0^2 = \frac{1}{(n_0-1)}\sum_{i=1}^{n_0}\left(X_i - \bar{X}_0\right)^2.$$

Then take $N - n_0$ further obsevations, with N being the smallest integer greater than n_0 and greater than or equal to

$$\left[s_0 t_{n_0-1}\left(1-\tfrac{1}{2}\alpha\right)/d\right]^2.$$

Show that, although N is random, $\sqrt{N}\left(\bar{X}-\mu\right)/s_0$, with $\bar{X} = \sum_{i=1}^{N} X_i /\!/ N$, has a $\mathcal{T}_{n_0-1}$ distribution. It follows that

$$\left[\bar{X} - s_0 t_{n_0-1}\left(1-\tfrac{1}{2}\alpha\right)/\sqrt{N}, \bar{X} + s_0 t_{n_0-1}\left(1-\tfrac{1}{2}\alpha\right)/\sqrt{N}\right]$$

is a confidence interval with confidence coefficient $(1-\alpha)$ for μ of length at most $2d$. (The sticky point of this approach is that we have no control over N, and, if σ is large, we may very likely be forced to take a prohibitively large number of observations. The reader interested in pursuing the study of sequential procedures such as this one is referrred to the book of Wetherill and Glazebrook, 1986, and the fundamental monograph of Wald, 1947.)

(Hint: Note that $\bar{X} = \frac{n_0}{N}\bar{X}_{n_0} + \frac{1}{N}\sum_{i=n_0+1}^{N} X_i$. By Theorem B.3.3, s_{n_0} is independent of $\bar{X}_{n_0}$. Because N depends only on s_{n_0}, given $N = k, \bar{X}$ has a $N(\mu, \sigma^2/k)$ distribution. Hence, $\sqrt{N}\left(\bar{X}-\mu\right)$ has a $N(0, \sigma^2)$ distribution and is independent of s_{n_0}.)

Solution:
As $\bar{X} = \frac{n_0}{N}\bar{X}_{n_0} + \frac{1}{N}\sum_{i=n_0+1}^{N} X_i$, and by Theorem B.3.3, s_{n_0} is independent of $\bar{X}_{n_0}$.
From the rule used to decide N, we see that N depends only on s_{n_0}.
Since $\frac{\sqrt{N}\left(\bar{X}-\mu\right)}{\sigma}$ ~ $N(0,1)$ distribution, and $\frac{(n_0-1)s_{n_0}^2}{\sigma^2}$ ~ $\chi^2_{n_0-1}$, then
$\Rightarrow \frac{\frac{\sqrt{N}\left(\bar{X}-\mu\right)}{\sigma}}{\sqrt{\frac{s_{n_0}^2}{\sigma^2}}} = \frac{\sqrt{N}\left(\bar{X}-\mu\right)}{s_{n_0}}$ ~ t_{n_0-1},
Hence, confidence interval with confidence coefficient $(1-\alpha)$ for μ of length at most $2d$ is $\left(\bar{X} - s_0 t_{n_0-1}\left(1-\frac{1}{2}\alpha\right)/\sqrt{N}, \bar{X} + s_0 t_{n_0-1}\left(1-\frac{1}{2}\alpha\right)/\sqrt{N}\right)$.

(Problem 4.5.4)
(a) Find c such that δ_c of Problem 4.1.1 has size α for $H : \theta \le \theta_0$.
(b) Derive the level $(1-\alpha)$ LCB corresponding to δ_c of part (a).
(c) Similarly derive the level $(1-\alpha)$ UCB for this problem and exhibit the confidence intervals obtained by putting two such bounds of level $(1-\alpha_1)$ and $(1-\alpha_2)$ together.
(d) Show that $[M_n, M_n/\alpha^{1.n}]$ is the shortest such confidence interval.
Solution:
(a) The power function is $\beta(\theta) = P_\theta(\delta_c(x) = 1) = 1 - \left(\frac{c}{\theta}\right)^n$, which is a monotone function in θ. Since δ_c has size α for $H : \theta \le \theta_0$, then

$$\alpha = \underset{\theta\le\theta_0}{Sup} P_\theta(\delta_c(x) = 1) = 1 - \left(\frac{c}{\theta_0}\right)^n,$$

$$\Rightarrow c = (1-\alpha)^{\frac{1}{n}}\theta_0$$

(b) $\delta_c(x) = \begin{cases} 1 & if\ M_n \ge c \\ 0 & otherwise \end{cases}$
$\Rightarrow$ The accept region is $A(\theta) = \{x : M_n < c\}$

$$= \left\{x : M_n < (1-\alpha)^{\frac{1}{n}}\theta_0\right\}$$

Then, by the duality theorem,

$$S(t) = \left\{\theta \in \Theta : \theta_0 > M_n(1-\alpha)^{-\frac{1}{n}}\right\}$$

$\Rightarrow$ The $(1-a)$ LCB of θ is $M_n(1-\alpha)^{-\frac{1}{n}}$.
(c) From $P_{\theta_0}(M_n \ge c) = 1 - \left(\frac{c}{\theta_0}\right)^n = 1-\alpha, \Rightarrow c = \theta_0(\alpha)^{\frac{1}{n}}$,
Then, $M_n \ge \theta_0(\alpha)^{\frac{1}{n}} \Rightarrow \theta_0 \le M_n(\alpha)^{-\frac{1}{n}}$
So the $(1-\alpha)$ UCB is $M_n(\alpha)^{-\frac{1}{n}}$.
Further, $\left(\frac{M_n}{(1-\alpha_1)^{\frac{1}{n}}}, \frac{M_n}{(\alpha_2)^{\frac{1}{n}}}\right)$ is the $1-\alpha_1-\alpha_2$ confidence interval by choosing α_1, α_2 properly, we can get the desired $(1-\alpha)$ confidence interval by choosing $\alpha_1 + \alpha_2 = \alpha$.

(d) The length of interval $\left(\frac{M_n}{(1-\alpha_1)^{\frac{1}{n}}}, \frac{M_n}{(\alpha-\alpha_1)^{\frac{1}{n}}}\right)$ is $\frac{M_n}{(1-\alpha_1)^{\frac{1}{n}}} - \frac{M_n}{(\alpha-\alpha_1)^{\frac{1}{n}}} = M_n\left(\frac{1}{(1-\alpha_1)^{\frac{1}{n}}} - \frac{1}{(\alpha-\alpha_1)^{\frac{1}{n}}}\right)$.
Since $\frac{\partial}{\partial\alpha_1}\left(\frac{1}{(1-\alpha_1)^{\frac{1}{n}}} - \frac{1}{(\alpha-\alpha_1)^{\frac{1}{n}}}\right) = -(1-\alpha_1)^{-1-\frac{1}{n}} + (\alpha-\alpha_1)^{-1-\frac{1}{n}} > 0$,

$\Rightarrow M_n\left(\frac{1}{(1-\alpha_1)^{\frac{1}{n}}}-\frac{1}{(\alpha-\alpha_1)^{\frac{1}{n}}}\right)$ is minimized when $\alpha_1=0$.
So, the above length of the interval is shortest when $\alpha_1=0, \alpha_2=\alpha$, when the interval is $\left(M_n, M_n/\alpha^{\frac{1}{n}}\right)$.

(Problem 4.7.1)
(a) Show that if θ has a beta, $\beta(r,s)$, distribution with r and s positive integers, then $\lambda = s\theta/r(1-\theta)$ has the F distribution $\mathcal{F}_{2r,2s}$.
Hint: See Sections B.2 and B.3.
(b) Suppose that given $\theta=\theta, X$ has a binomial, $\mathcal{B}(n,\theta)$, distribution and that θ has beta, $\beta(r,s)$ distribution with r and s integers. Show how the quantiles of the F distribution can be used to find upper and lower credible bounds for λ and for θ.
Solution:
(a) Let $X\tilde{}Gamma\left(r,\frac{1}{2}\right), Y\tilde{}Gamma\left(s,\frac{1}{2}\right)$,
then by theorem B.2.3, $\theta\hat{=}\frac{X}{X+Y}\tilde{}Beta(r,s)$
Since $X/2r\tilde{}\chi^2_{2r}, Y/2s\tilde{}\chi^2_{2s}$, then $\frac{X/2r}{Y/2s}\tilde{}F_{2r,2s}$, i.e. $\frac{sX}{rY}\tilde{}F_{2r,2s}$.
And $\theta=\frac{X}{X+Y}\Rightarrow\frac{\theta}{1-\theta}=\frac{X}{Y}$, and thus $\lambda=\frac{s\theta}{r(1-\theta)}\tilde{}F_{2r,2s.}$
(b) $g(\theta|x)=f(x|\theta)\pi(\theta)\propto\theta^x(1-\theta)^{n-x}\theta^{r-1}(1-\theta)^{s-1}$
$\Rightarrow\theta|X\tilde{}Beta(x+r,n+s-x)$
As $P\left(F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}\le\frac{(n+s-x)\theta}{(x+r)(1-\theta)}\le F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}\right)=1-\alpha$
$\Rightarrow P\left(\frac{(x+r)s}{(n+s-x)r}F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}\le\lambda\le\frac{(x+r)s}{(n+s-x)r}F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}\right)=1-\alpha$
The upper and lower credible bounds for $\lambda|x$ is

$$\left(\frac{(x+r)s}{(n+s-x)r}F_{2(x+r),2(n+s-x),\frac{\alpha}{2}},\frac{(x+r)s}{(n+s-x)r}F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}\right)$$

And
$\Rightarrow P\left(\frac{(x+r)}{(n+s-x)}F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}+1\le\frac{1}{(1-\theta)}\le\frac{(x+r)}{(n+s-x)}F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}+1\right)=1-\alpha$
$\Rightarrow P\left(\frac{(x+r)F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}+(n+s-x)}\le\theta\le\frac{(x+r)F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}+(n+s-x)}\right)=1-\alpha$
The upper and lower credible bounds for $\theta|x$ is

$$\left(\frac{(x+r)F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n+s-x),\frac{\alpha}{2}}+(n+s-x)},\frac{(x+r)F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n+s-x),1-\frac{\alpha}{2}}+(n+s-x)}\right)$$

(Problem 4.8.4)
Suppose that given $\theta=\theta$, X is a binomial, $\mathcal{B}(n,\theta)$, random variable, and that θ has a beta, $\beta(r,s)$, distribution. Suppose that Y, which is not observable, has a $B(m,\theta)$ distribution given $\theta=\theta$. Show that the conditional (predictive) distrbibution of Y given $X=x$ is

$$q(y|x)=\binom{m}{y}B(r+x+y,s+n-x+m-y)/B(r+x,s+n-x)$$

where $B(\cdot,\cdot)$ denotes the beta function. (This $q(y|x)$ is sometimes called the Polya distribution.)
Hint: First show that

$$q(y|x)=\int p(y|\theta)\pi(\theta|x)\,d\theta.$$

Solution:

Since $Y|\theta\tilde{\ }Binomial\,(m,\theta)\,,\theta\tilde{\ }Beta\,(r,s)\,,\&\ x|\theta\tilde{\ }Binomial\,(n,\theta)$

$\Rightarrow \theta|x\tilde{\ }Beta\,(r+x,n+s-x)$

Then,

$q\,(y|x) = \int p\,(y,\theta|x)\,d\theta = \int p\,(y|\theta)\,\pi\,(\theta|x)\,d\theta$

$= \int \binom{m}{y}\theta^{y}\,(1-\theta)^{m-y}\,\frac{1}{Beta(r+x,n+s-x)}\theta^{r+x-1}\,(1-\theta)^{n+s-x-1}\,d\theta$

$= \frac{\binom{m}{y}}{Beta(r+x,n+s-x)}\int \theta^{y+r+x-1}\,(1-\theta)^{n+s+m-x-y-1}\,d\theta$

$= \binom{m}{y}\frac{Beta(x+y+r,m+n+s-x-y)}{Beta(r+x,n+s-x)}.$