Homework 8 Solution

1. If $X\tilde{}Poisson(\lambda)$ and a Bayesian uses a prior for λ that is Exponential with mean 1, what is the Bayesian's (0-1 loss) Bayes test of $H_0: \lambda \le 2$ versus $H_1: \lambda > 2$?

Solution:
For prior $g(\lambda) = e^{-\lambda} I(\lambda > 0)$, the posterior distribution is

$$g(\lambda|x) \propto \frac{\frac{e^{-\lambda}\lambda^x}{x!} e^{-\lambda}}{\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} e^{-\lambda} d\lambda}$$

$\Rightarrow \lambda|x\tilde{}Gamma\left(x+1, \frac{1}{2}\right)$.

Then the Bayesian's (0-1 loss) Bayes test is:

$$\phi(x) = \begin{cases} 1 & P(\lambda \le x) < 0.5 \\ 0 & o.w. \end{cases}$$

where, $P(\lambda \le 2) = \frac{1}{\Gamma(x+1)\left(\frac{1}{2}\right)^{x+1}} \int_0^2 \lambda^x e^{-2\lambda} d\lambda.$

2. If $X\tilde{}Binomial(5, p)$ and a Bayesian uses a prior for p that is $Beta(2,1)$, what is the Bayesian's (0-1 loss) Bayes test of $H_0: 0.4 \le p \le 0.6$ versus $H_0: p < 0.4$ or $p > 0.6$?
Solution:

For prior $g(p) = \frac{1}{B(2,1)} p^1 (1-p)^0 I(0 < p < 1) = 2pI(0 < p < 1)$, the posterior distribution is

$$g(p|x) = \frac{\binom{5}{x} p^x (1-p)^{5-x} 2p}{\int_0^1 \binom{5}{x} p^x (1-p)^{5-x} 2p dp} = \frac{1}{B(x+2, 6-x)} p^{x+1} (1-p)^{5-x}$$

$\Rightarrow p|x\tilde{}Beta(x+2, 6-x)$

Then the Bayesian's (0-1 loss) Bayes test is

$$\phi(x) = \begin{cases} 1 & P(0.4 \le p \le 0.6) < 0.5 \\ 0 & o.w. \end{cases}$$

where, $P(0.4 \le p \le 0.6) = \frac{1}{B(x+2,6-x)} \int_{0.4}^{0.6} p^{x+1} (1-p)^{5-x} dp.$

3.
(Problem 4.3.1)
Let X_i be the number of arrivals at a service counter on the ith of a sequence of n days. A possible model for these data is to assume that customers arrive according to a homogeneous Poisson process and, hence, that the X_i are a sample from a Poisson distribution with parameter θ, the expected number of arrivals per day. Suppose that if $\theta \le \theta_0$ it is not worth keeping the counter open.
(a) Exhibit the optimal (UMP) test statistic for $H: \theta \le \theta_0$ versus $K: \theta > \theta_0$.
(b) For what levels can you exhibit a UMP test?
(c) What distribution tables would you need to calculate the power function of the UMP test?
Solution:

(a) $P(X = x|\theta) = \frac{1}{\prod x_i!} e^{-n\theta} \theta^{\sum_i x_i} = \frac{1}{\prod x_i!} \exp\left\{\sum_i x_i \log\theta - n\theta\right\}$

$\Rightarrow LR = \exp\left\{\sum_i x_i (\log\theta_2 - \log\theta_1) - n(\theta_2 - \theta_1)\right\}$
L is increasing in $T = \sum_i x_i$, and thus the poisson model is an MLR family in T.

So, the UMP test statistic is

$$\delta_k(x) = \begin{cases} 1 & \sum_i x_i > k \\ \gamma & \sum_i x_i = k \\ 0 & \sum_i x_i < k \end{cases}$$

(b) Consider $P\left(\sum_i x_i > k|\theta_0\right) + \gamma P\left(\sum_i x_i = k|\theta_0\right) = \alpha$, we see that by choosing γ appropriately, any level UMP test can be exhibited.
(c) As

$$\frac{\sum_i x_i - n\theta}{\sqrt{n\theta}} \sim N(0,1),$$

we can use normal table to calculate the power function of the UMP test.

(Problem 4.3.2)
Consider the foregoing situation of Problem 4.3.1. You want to ensure that if the arrival rate is ≤ 10, the probability of your deciding to stay open is ≤ 0.01, but if the arrival rate is ≥ 15, the probability of your deciding to stay open is ≤ 0.01. How many days must you observe to ensure that the UMP test of Problem 4.3.1 achieves this? (Use the normal approximation.)
Solution:

As $\frac{\sum_i x_i - 10n}{\sqrt{10n}} \sim N(0,1)$, and $P(\theta \leq 10) \leq 0.01$,
we can construct our test as

$$\phi(x) = \begin{cases} 1 & \sum_i x_i > 10n + Z_{1-0.01}\sqrt{10n} \\ 0 & o.w. \end{cases}.$$

And also $P_{\theta_1=15}\left(\sum_i x_i > 10n + Z_{0.99}\sqrt{10n}\right) \leq 0.01$

$$\Rightarrow 10n + Z_{0.99}\sqrt{10n} \leq 15n - Z_{0.99}\sqrt{15n}$$
$$\Rightarrow 10n + 2.326\sqrt{10n} \leq 15n - 2.326\sqrt{15n}$$
$$\Rightarrow n \geq 10.71$$

Then, take $n = 11$.

(Problem 4.3.5)
Show that if $X_1, ..., X_n$ is a sample from a truncated binomial distribution with

$$p(x,\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} / \left[1 - (1-\theta)^n\right], x = 1, ..., n,$$

then $\sum_{i=1}^n X_i$ is an optimal tetst statistic for testing $H : \theta = \theta_0$ versus $K : \theta > \theta_0$.
Solution:

$$p\left(\underset{\sim}{x}, \theta\right) = \prod_i \binom{n}{x_i} \theta^{\sum_i x_i} (1-\theta)^{\sum_i (n-x_i)} / [1-(1-\theta)^n]^n$$
$$\Rightarrow LR = \frac{\theta^{\sum_i x_i}(1-\theta)^{\sum_i (n-x_i)}/[1-(1-\theta)^n]^n}{\theta_0^{\sum_i x_i}(1-\theta_0)^{\sum_i (n-x_i)}/[1-(1-\theta_0)^n]^n}$$
$$= \left(\frac{\theta}{\theta_0}\right)^{\sum_i x_i} \left(\frac{1-\theta}{1-\theta_0}\right)^{\sum_i (n-x_i)} \left(\frac{1-(1-\theta_0)^n}{1-(1-\theta)^n}\right)^n.$$

L is increasing in $\sum_i x_i$, and thus the truncated binomial model is MLR family in T. So, $T = \sum_i x_i$ is an optimal test statistic for testing $H : \theta = \theta_0$ versus $K : \theta > \theta_0$.

(Problem 4.3.6)
Let $X_1, ..., X_n$ denote the incomes of n persons persons chosen at random from a certain population. Suppose that each X_i has the Pareto density

$$f(x,\theta) = c^{\theta}\theta x^{-(1+\theta)}, x > c$$
where $\theta > 1$ and $c > 0$.
(a) Express mean income μ in terms of θ.
(b) Find the optimal test statistic for testing $H : \mu = \mu_0$ versus $K : \mu > \mu_0$.
(c) Use the central limit theorem to find a normal approximation to the critical value of test in part (b).
Hint: Use the results of Theorem 1.6.2 to find the mean and variance of the optimal test statistic.

Solution:
(a) $EX = \int_c^\infty x c^{\theta}\theta x^{-(1+\theta)}dx = c^{\theta}\theta\int_c^\infty x^{-\theta}dx = c^{\theta}\theta \frac{x^{1-\theta}}{1-\theta}|_c^\infty = \frac{c\theta}{\theta-1}$
i.e. the mean income $\mu = \frac{c\theta}{\theta-1}$.
(b) By $\mu = \frac{c\theta}{\theta-1} \Rightarrow \theta = \frac{\mu}{\mu-c}$, then

$$f\left(\underset{\sim}{x}|\mu\right) = c^{\frac{n\mu}{\mu-c}}\left(\frac{\mu}{\mu-c}\right)^n \prod x_i^{-\left(1+\frac{\mu}{\mu-c}\right)},$$

$$\Rightarrow LR = \left(\frac{\mu(\mu_0-c)}{\mu_0(\mu-c)}\right)^n \exp\left\{\left(\frac{\mu}{\mu-c} - \frac{\mu_0}{\mu_0-c}\right)\left(n\ln c - \sum_i \log x_i\right)\right\}, \mu > \mu_0$$

As $\frac{\mu}{\mu-c} - \frac{\mu_0}{\mu_0-c} < 0$, L is increasing in $\sum_i \log x_i$, and the Pareto family is MLR in $T = \sum_i \log x_i$. So $\sum_i \log x_i$ can be used as an optimal test statistic for testing $H : \mu = \mu_0$ versus $K : \mu > \mu_0$.

(c) As $f\left(\underset{\sim}{x}|\theta\right) = \prod x_i^{-1}\exp\left\{-\theta\sum_i \log x_i + n\theta\log c + n\log\theta\right\}$,
By Theorem 1.6.2

$$\Rightarrow E\left(\sum_i \log x_i\right) = -n\left(\log c + \frac{1}{\theta}\right) = \mu_0$$
$$Var\left(\sum_i \log x_i\right) = \frac{n}{\theta^2} = \sigma_0^2$$

As $\frac{T(x)-\mu_0}{\sqrt{\sigma_0^2}} \sim N(0,1)$, the critical value is $\mu_0 + Z_{1-\frac{\alpha}{2}}\sqrt{\sigma_0^2}$.

(Problem 4.3.9)
Let $X_1, ..., X_n$ be i.i.d. with distribution function $F(x)$. We want to test whether F is exponential, $F(x) = 1 - \exp(-x), x > 0$, or Weibull, $F(x) = 1 - \exp\left(-x^{\theta}\right), x > 0, \theta > 0$. Find the MP test for testing $H : \theta_0 = 1$ versus $K : \theta = \theta_1 > 1$. Show that the test is not UMP.

Solution:
For exponential distribution, $f(x) = \exp(-x), x > 0$; and for Weibull distribution, $f(x) = \theta x^{\theta-1}\exp\left(-x^{\theta}\right), x > 0, \theta > 0$. The likelihood ratio is
$L(x,\theta_0,\theta_1) = \frac{f(x,\theta_1)}{f(x,\theta_0)} = \theta x^{\theta-1}\exp\left(-x^{\theta}+x\right), x > 0$.

For any $\theta_1 > 1$, by Neyman-Pearson Theorem, there exists an MP test for testing $H : \theta_0 = 1$ versus $K : \theta = \theta_1$ in the form of

$$\phi_1(x) = \begin{cases} 1 & L(x,\theta_0,\theta_1) \geq k_1 \\ 0 & L(x,\theta_0,\theta_1) < k_1 \end{cases}$$

Then for any other $\theta_2 > 1$, $\theta_2 \neq \theta_1$, there also exists an MP test as

$$\phi_2(x) = \begin{cases} 1 & L(x,\theta_0,\theta_2) \geq k_2 \\ 0 & L(x,\theta_0,\theta_2) < k_2 \end{cases}$$

If there exist a UMP test, then it should be MP test for both $K : \theta = \theta_1$

& $K : \theta = \theta_2$ simultaneously.
Since both likelihood ratio test statistics above are with θ involved, they can not be simultaneously MP for both tests. So there is no UMP test for $H : \theta_0 = 1$ versus $K : \theta > 1$.

(Problem 4.3.11)
Show that under the assumptions of Theorem 4.3.1 and 0-1 loss, every Bayes test for $H : \theta \leq \theta_0$ versus $K : \theta > \theta_1$ is of the form δ_t for some t.
Hint: A Bayes test rejects (accepts) H if $\int_{\theta_1}^{\infty} p(x,\theta)\, d\pi(\theta) \,/\, \int_{-\infty}^{\theta_0} p(x,\theta)\, d\pi(\theta) \overset{>}{(<)} 1.$
The left-hand side equals $\frac{\int_{\theta_1}^{\infty} L(x,\theta,\theta_0) d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta,\theta_0) d\pi(\theta)}$.
The numerator is an increasing function of $T(x)$, the denominator decreasing.

Solution:

The Bayesian's (0-1 loss) Bayes test is in the form of
$$\phi(x) = \begin{cases} 1 & \frac{P(\theta>\theta_1|x)}{P(\theta\leq\theta_0|x)} > 1 \\ 0 & o.w. \end{cases}$$
As
$$\frac{P(\theta>\theta_1|x)}{P(\theta\leq\theta_0|x)} = \frac{\int_{\theta_1}^{\infty} P(\theta|x)d\theta}{\int_{-\infty}^{\theta_0} P(\theta|x)d\theta} = \frac{\int_{\theta_1}^{\infty} L(x,\theta)d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta)d\pi(\theta)} = \frac{\int_{\theta_1}^{\infty} \frac{L(x,\theta)}{L(x,\theta_0)}d\pi(\theta)}{\int_{-\infty}^{\theta_0} \frac{L(x,\theta)}{L(x,\theta_0)}d\pi(\theta)} = \frac{\int_{\theta_1}^{\infty} L(x,\theta,\theta_0)d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta,\theta_0)d\pi(\theta)},$$
$$\Rightarrow \phi(x) = \begin{cases} 1 & \frac{\int_{\theta_1}^{\infty} L(x,\theta,\theta_0)d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta,\theta_0)d\pi(\theta)} > 1 \\ 0 & o.w. \end{cases}$$
And since the numerator of $\frac{\int_{\theta_1}^{\infty} L(x,\theta,\theta_0)d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta,\theta_0)d\pi(\theta)}$ is an increasing function of $T(x)$, and the denomintor of $\frac{\int_{\theta_1}^{\infty} L(x,\theta,\theta_0)d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta,\theta_0)d\pi(\theta)}$ is a decreasing function of $T(x)$, then the above test equals
$$\phi(x) = \begin{cases} 1 & T(x) > t \\ 0 & o.w. \end{cases}.$$

4.(Optional)
Prove the following "folling-in" lemma:
Suppose that g_0 and g_1 are two distinct, positive probability densities defines on an interval in $\Re^1$. If the ratio g_1/g_0 is nondecreasing in a real-valued function $T(x)$, then the family of densities $\{g_\alpha | \alpha \in [0,1]\}$ for $g_\alpha = \alpha g_1 + (1-\alpha) g_0$ has the MLR property in $T(x)$.

Solution:
As for $\forall \alpha_1, \alpha_2 \in [0,1], \alpha_2 > \alpha_1$,
$$\frac{g_{\alpha_2}}{g_{\alpha_1}} = \frac{\alpha_2 g_1 + (1-\alpha_2) g_0}{\alpha_1 g_1 + (1-\alpha_1) g_0} = \frac{\alpha_2 \frac{g_1}{g_0} + (1-\alpha_2)}{\alpha_1 \frac{g_1}{g_0} + (1-\alpha_1)},$$
then
$$\frac{d}{dT}\left(\frac{g_{\alpha_2}}{g_{\alpha_1}}\right) = \frac{(*)}{\left(\alpha_1 \frac{g_1}{g_0} + (1-\alpha_1)\right)^2},$$
where,
$$(*) = \alpha_2 \frac{d}{dT}\left(\frac{g_1}{g_0}\right)\left[\alpha_1 \frac{g_1}{g_0} + (1-\alpha_1)\right] - \alpha_1 \frac{d}{dT}\left(\frac{g_1}{g_0}\right)\left[\alpha_2 \frac{g_1}{g_0} + (1-\alpha_2)\right]$$
$$= \left[\alpha_2(1-\alpha_1) - \alpha_1(1-\alpha_2)\right] \frac{d}{dT}\left(\frac{g_1}{g_0}\right)$$

$= [\alpha_2 - \alpha_1] \frac{d}{dT}\left(\frac{g_1}{g_0}\right) \geq 0$

$\Rightarrow \frac{d}{dT}\left(\frac{g_{\alpha_2}}{g_{\alpha_1}}\right) \geq 0$

$\Rightarrow g_\alpha$ is MLR in $T(x)$.

5. (Optional)

Two possible definitions of "UMP size α" are:

Definition 1 A test ϕ of $H_0 : \theta \in \Theta_0$ vs. $H_a : \theta \in \Theta_1$ is UMP of size α provided

(i) it is of size α, and

(ii) for any other test ϕ' of size α, $\pi_\phi(\theta) \geq \pi_{\phi'}(\theta), \forall\theta \in \Theta_1$.

Definition 2...as in Definition 1, except in ii), let ϕ' be of size $\leq \alpha$.

At first glance, it may seem that Definition 1 is weaker than Definition 2 (it might appear that ϕ could satisfy Definition 1 and fail to satisfy Definition 2). But, in fact, these two definitions are equivalent. Show the equivalence.

(Hint: If ϕ were to satisfy Definition 1 but not Definition 2, there would need to be a test ϕ' with $\alpha' = \sup_{\theta\in\Theta_0} \pi_{\phi'}(\theta) < \alpha$, such that for some $\theta^* \in \Theta_1 > \pi_\phi(\theta^*)$. Consider the test $\phi''(x) = \left(\frac{1-\alpha}{1-\alpha'}\right)\phi'(x) + \left(1 - \left(\frac{1-\alpha}{1-\alpha'}\right)\right)1$.)

Solution:

Suppose $\exists\ \phi$ satisfying Definition 1 but not Definition 2, there should $\exists$ a test ϕ' with $\alpha' = \sup_{\theta\in\Theta_0} \pi_{\phi'}(\theta) < \alpha$ such that for some $\theta^* \in \Theta_1$, $\pi_{\phi'}(\theta^*) > \pi_\phi(\theta^*)$.

Consider the test $\phi''(x) = \left(\frac{1-\alpha}{1-\alpha'}\right)\phi'(x) + \left(1 - \left(\frac{1-\alpha}{1-\alpha'}\right)\right)1$.

The size of ϕ'' is

$\alpha'' = \sup_{\theta\in\Theta_0} \pi_{\phi''}(\theta) = \left(\frac{1-\alpha}{1-\alpha'}\right)\sup_{\theta\in\Theta_0} \pi_{\phi'}(\theta) + \left(1 - \left(\frac{1-\alpha}{1-\alpha'}\right)\right)1$

$= \left(\frac{1-\alpha}{1-\alpha'}\right)\alpha' + \left(1 - \left(\frac{1-\alpha}{1-\alpha'}\right)\right) = 1 - \left(\frac{1-\alpha}{1-\alpha'}\right)(1-\alpha') = \alpha$

And,

$\pi_{\phi''}(\theta^*) = P_{\theta^*}(\phi''(x) = 1)$

$= P_{\theta^*}\left(\left(\frac{1-\alpha}{1-\alpha'}\right)\phi'(x) + \left(1 - \left(\frac{1-\alpha}{1-\alpha'}\right)\right)1 = 1\right)$

$= P_{\theta^*}(\phi'(x) = 1)$

$= \pi_{\phi'}(\theta^*) > \pi_\phi(\theta^*)$.

So, ϕ'' is of size α and $\exists\theta^* \in \Theta_1, s.t. \pi_{\phi''}(\theta^*) > \pi_\phi(\theta^*)$, which is contradictive to the fact that ϕ satisfies Definition 1. Therefore, the two definitions are equivalent.