STAT 543 Homework 2 Solution

1.
(Problem 1.5.1)
Let $X_1,\dots,X_n$ be a sample from a Poisson, $\mathrm{P}(\theta)$, population where $\theta > 0$.

(a) Show directly that $\sum_{i=1}^n X_i$ is sufficient for θ.
(b) Establish the same result using the factorization theorem.

Solution:
(a)
Since $X_1,\dots,X_n \underset{\sim}{i.i.d}\ Poisson(\theta)$,

$$P(X_1 = x_1,\dots,X_n = x_n \mid \theta)$$
$$= \prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!} I(x_i \geq 0)$$
$$= \frac{e^{-n\theta}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \prod_{i=1}^n I(x_i \geq 0)$$

And also $\sum_{i=1}^n X_i \sim Poisson(n\theta)$, then

$$P(\sum_{i=1}^n X_i = t \mid \theta) = \frac{e^{-n\theta}(n\theta)^t}{t!}.$$

So

$$P(X_1 = x_1,\dots,X_n = x_n \mid \sum_{i=1}^n X_i = t, \theta)$$
$$= \frac{P(X_1 = x_1,\dots,X_n = x_n, \sum_{i=1}^n X_i = t \mid \theta)}{P(\sum_{i=1}^n X_i = t \mid \theta)}$$
$$= \frac{e^{-n\theta}\theta^t \prod_{i=1}^n I(x_i \geq 0) / \prod_{i=1}^n x_i!}{e^{-n\theta}(n\theta)^t / t!}$$
$$= \frac{t! \prod_{i=1}^n I(x_i \geq 0)}{n^t \prod_{i=1}^n x_i!}$$

, which is independent on θ. So, $\sum_{i=1}^n X_i$ is sufficient for θ.

(b)
Since $\quad P(X_1 = x_1,\dots,X_n = x_n \mid \theta)$

$$= \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} I(x_i \geq 0)$$

$$= \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \prod_{i=1}^{n} I(x_i \geq 0)$$

Let $T(\underline{X}) = \sum_{i=1}^{n} X_i$, then

$$P(X_1 = x_1, ..., X_n = x_n \mid \theta)$$
$$= g(T(\underline{x}) \mid \theta) h(\underline{x})$$

, where

$$g(T(\underline{x}), \theta) = e^{-n\theta}\theta^{T(\underline{x})}$$

, and

$$h(\underline{x}) = \frac{\prod_{i=1}^{n} I(x_i \geq 0)}{\prod_{i=1}^{n} x_i!}$$

, which is independent of θ.

Then by the factorization theorem, $\sum_{i=1}^{n} X_i$ is sufficient for θ.

2.

(Problem 1.5.2)

Let n items be drawn in order without replacement from a shipment of N items of which $N\theta$ are bad. Let $X_i = 1$ if the i^{th} item drawn is bad, and =0 otherwise.

Show that $\sum_{i=1}^{n} X_i$ is sufficient for θ directly and by the factorization theorem.

<u>Solution:</u>

Since $P(X_1 = x_1, ..., X_n = x_n \mid \sum_{i=1}^{n} X_i = t) = \frac{1}{\binom{n}{t}}$, where $X_i = \begin{cases} 1 & if \quad X_i \quad is \quad bad \\ 0 & if \quad X_i \quad is \quad good \end{cases}$;

, which is free of θ, then $T = \sum_{i=1}^{n} X_i$ is a sufficient statistics of θ

On the other hand, $\sum_{i=1}^{n} X_i$ is in a hyper-geometric distribution, so

$$P(\sum_{i=1}^{n} X_i = t \mid \theta)$$

$$= \frac{\binom{N\theta}{t}\binom{N-N\theta}{n-t}}{\binom{N}{n}}$$

Then,

$$\begin{aligned} &P(X_1 = x_1, ..., X_n = x_n) \\ &= P(X_1 = x_1, ..., X_n = x_n, \textstyle\sum_{i=1}^n X_i = t) \\ &= P(X_1 = x_1, ..., X_n = x_n \mid \textstyle\sum_{i=1}^n X_i = t) P(\textstyle\sum_{i=1}^n X_i = t) \\ &= \frac{1}{\binom{n}{t}} \frac{\binom{N\theta}{t}\binom{N-N\theta}{n-t}}{\binom{N}{n}} \\ &= g(T(\underline{x}), \theta) h(\underline{x}) \end{aligned}$$

, where $g(T(\underline{x}), \theta) = \frac{1}{\binom{n}{t}} \frac{\binom{N\theta}{t}\binom{N-N\theta}{n-t}}{\binom{N}{n}}$, and $h(\underline{x}) = 1$.

So, by factorization theorem, $T = \sum_{i=1}^n X_i$ is a sufficient statistics of θ.

3.

(Problem 1.5.3)

Suppose $X_1, \ldots, X_n$ is a sample from a population with one of the following densities.

a). $p(x,\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0$. This is beta, $\beta(\theta,1)$, density.

b). $p(x,\theta) = \theta a x^{a-1} \exp(-\theta x^a), x > 1, \theta > 0, a > 0$. This is known as the *Weibull* Density.

c). $p(x,\theta) = \theta a^\theta / x^{(\theta+1)}, x > a, \theta > 0, a > 0$. This is known as the *Parto* density.

In all cases, find the real-value sufficient statistic for θ and a fixed.

<u>Solution:</u>

a) Since

$$P(X_1 = x_1, ..., X_n = x_n \mid \theta) = \prod_{i=1}^n P(X_i = x_i \mid \theta) = \prod_{i=1}^n \theta x_i^{\theta-1} I(0 < x_i < 1)$$

$$= \theta^n \prod_{i=1}^n x_i^{\theta-1} I(0 < x_i < 1)$$

Let

$$T(\underline{X}) = \prod_{i=1}^n X_i$$

, then

$$P\left(X_1 = x_1, \ldots, X_n = x_n \mid \theta\right) = g\left(T\left(\underline{x}\right) \mid \theta\right) h(\underline{x})$$

, where

$$g\left(T\left(\underline{x}\right) \mid \theta\right) = \theta^n T\left(\underline{x}\right)^{\theta-1}$$

, and

$$h(\underline{x}) = \prod_{i=1}^{n} I(0 < x_i < 1)$$

Then by factorization theorem, $T(\underline{X}) = \prod_{i=1}^{n} X_i$ is sufficient for θ.

b) Since

$$\begin{aligned} &P\left(X_1 = x_1, \ldots, X_n = x_n \mid \theta\right) \\ &= \prod_{i=1}^{n} \theta a x_i^{a-1} \exp\left(-\theta x_i^a\right) I\left(x_i > 0\right) \\ &= \theta^n a^n \left(\prod_{i=1}^{n} x_i\right)^{a-1} \exp\left(-\theta \sum x_i^a\right) \left(\prod_{i=1}^{n} I\left(x_i > 0\right)\right) \end{aligned}$$

Let

$$T(\underline{X}) = \sum X_i^a$$

, then

$$P\left(X_1 = x_1, \ldots, X_n = x_n \mid \theta\right) = g\left(T\left(\underline{x}\right) \mid \theta\right) h(\underline{x})$$

, where

$$g\left(T\left(\underline{x}\right) \mid \theta\right) = \theta^n a^n \exp\left(-\theta T\left(\underline{x}\right)\right)$$

, and

$$h(\underline{x}) = \left(\prod_{i=1}^{n} x_i\right)^{a-1} \prod_{i=1}^{n} I(x_i > 1).$$

Then by factorization theorem, $T(\underline{X}) = \sum X_i^a$ is sufficient for θ.

c).

$$\begin{aligned} &P\left(X_1 = x_1, \ldots, X_n = x_n \mid \theta\right) \\ &= \prod_{i=1}^{n} \theta a^{\theta} \big/ x_i^{(\theta+1)} I\left(x_i > a\right) \\ &= \frac{\theta^n a^{n\theta}}{\prod_{i=1}^{n} x_i^{\theta+1}} \prod_{i=1}^{n} I\left(x_i > a\right) \end{aligned}$$

Let

$$T(\underline{X}) = \prod_{i=1}^{n} X_i$$

, then

$$P\left(X_1 = x_1, \ldots, X_n = x_n \mid \theta\right) = g\left(T\left(\underline{x}\right) \mid \theta\right) h(\underline{x})$$

, where

$$g\left(T\left(\underline{x}\right)|\theta\right)=\frac{\theta^n a^{n\theta}}{T\left(\underline{x}\right)^{\theta+1}}$$

, and

$$h(\underline{x})=\prod_{i=1}^{n} I(x_i>a)\,.$$

Then by factorization theorem, $T(\underline{X})=\prod_{i=1}^{n} X_i$ is sufficient for θ.

4.

(Problem 1.5.7)

Let $X_1,\dots,X_n$ be a sample form a population with density $p(x,\theta)$ given by

$$p(x,\theta)=\begin{cases}\frac{1}{\sigma}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\} & ,if \quad x\geq\mu \\ 0 & otherwise\end{cases}$$

Here $\theta=\left(\mu,\sigma\right)$ with $-\infty<\mu<\infty \quad ,\sigma>0$.

a) Show that $\min(X_1,\dots,X_n)$ is sufficient for μ when σ is fixed.
b) Find a one-dimensional sufficient statistic for σ when is μ fixed.
c) Exhibit a two-dimensional sufficient statistic for θ.

Solutions:

a)

$$\begin{aligned}&P\left(X_1=x_1,\dots,X_n=x_n\,|\,\theta\right)\\&=\prod_{i=1}^{n}\frac{1}{\sigma}\exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right]I\left(x_i\geq\mu\right)\\&=\frac{1}{\sigma^n}\exp\left[-\left(\frac{\sum x_i-n\mu}{\sigma}\right)\right]\prod_{i=1}^{n}I\left(x_i\geq\mu\right)\end{aligned}$$

Let

$$T(\underline{X})=\min\left(X_1,\dots,X_n\right)=X_{(1)}$$

, then

$$P\left(X_1=x_1,\dots,X_n=x_n\,|\,\theta\right)=g\left(T\left(\underline{x}\right)\right)h(\underline{x})$$

, where

$$g\left(T(\underline{x})\right)=\frac{1}{\sigma^n}\exp\left(\frac{n\mu}{\sigma}\right)I\left(T(\underline{x})\geq\mu\right)$$

, and

$$h(\underline{x})=\exp\left(\frac{-\sum x_i}{\sigma}\right),\ \sigma \text{ is fixed.}$$

Then by factorization theorem, $T(\underline{X}) = \min\left(X_1, ..., X_n\right) = X_{(1)}$ is sufficient for μ when σ is fixed.

b).

Let

$$T(\underline{X}) = \prod_{i=1}^{n} X_i$$

, then

$$P\left(X_1 = x_1, ..., X_n = x_n \mid \theta\right) = g\left(T\left(\underline{x}\right)\right) h(\underline{x})$$

, where

$$g\left(T(\underline{x})\right) = \frac{1}{\sigma^n} \exp\left(-\frac{T(\underline{x}) - n\mu}{\sigma}\right)$$

, and

$$h(\underline{x}) = \prod_{i=1}^{n} I\left(x_i \geq \mu\right).$$

Then by factorization theorem, $T(\underline{X}) = \prod_{i=1}^{n} X_i$ is sufficient for σ when is μ fixed.

c).

Let

$$T(\underline{X}) = \left(T_1(\underline{X}), \quad T_2(\underline{X})\right) = \left(\sum X_i, X_{(1)}\right)$$

, then

$$P\left(X_1 = x_1, ..., X_n = x_n \mid \theta\right) = g\left(T\left(\underline{x}\right)\right) h(\underline{x})$$

, where

$$g\left(T(\underline{x})\right) = \frac{1}{\sigma^n} \exp\left(-\frac{T_1(\underline{x}) - n\mu}{\sigma}\right) I\left(T_2(\underline{x}) \geq \mu\right)$$

, and

$$h(\underline{x}) = 1.$$

Then by factorization theorem,

$$T(\underline{X}) = \left(T_1(\underline{X}), \quad T_2(\underline{X})\right) = \left(\sum X_i, X_{(1)}\right)$$

is sufficient statistics for $\theta = \left(\mu, \sigma\right)$.

5.

(Problem 1.5.9)

Let $X_1, \ldots, X_n$ be a sample from a population with density

$$f_\theta(x) = \begin{cases} a(\theta) h(x) & if\, \theta_1 \leq x \leq \theta_2 \\ 0 & otherwise \end{cases}$$

, where $h(x) \geq 0, \theta = (\theta_1, \theta_2)$ with $-\infty < \theta_1 \leq \theta_2 < \infty$, and $a(\theta = \left[\int_{\theta_1}^{\theta_2} h(x)dx\right]^{-1}$ is assumed to exist. Find a two-dimensional sufficient statistic for this problem and apply your results to the $u[\theta_1, \theta_2]$ family of distributions.

Solutions:

$$f_\theta(x) = \prod_{i=1}^{n} a(\theta) h(x_i) I(\theta_1 \leq x_i \leq \theta_2)$$
$$= a(\theta)^n \prod_{i=1}^{n} h(x_i) I(\theta_1 \leq x_i \leq \theta_2)$$

Let

$$T(\underline{X}) = \left(T_1(\underline{X}), \quad T_2(\underline{X})\right) = \left(\min_{1\leq i\leq n} X_i, \max_{1\leq i\leq n} X_i\right) = (X_{(1)}, X_{(n)})$$

, then

$$f_\theta(x) = g\left(T(\underline{x})\right) h(\underline{x})$$

, where

$$g\left(T(\underline{x})\right) = a(\theta)^n\, I\left(\theta_1 \leq T_1(\underline{x}) \leq \theta_2\right) I\left(\theta_1 \leq T_2(\underline{x}) \leq \theta_2\right)$$

, and

$$h(\underline{x}) = \prod_{i=1}^{n} h(x_i).$$

Then by factorization theorem,

$$T(\underline{X}) = \left(T_1(\underline{X}), \quad T_2(\underline{X})\right) = \left(\min_{1\leq i\leq n} X_i, \max_{1\leq i\leq n} X_i\right) = (X_{(1)}, X_{(n)})$$

is sufficient statistics for $\theta = (\theta_1, \theta_2)$.

Especially for $u[\theta_1, \theta_2]$,

$$a(\theta) = \frac{1}{\theta_2 - \theta_1}$$

, and $h(x) = 1$.

So

$$T(\underline{X}) = \left(T_1(\underline{X}), \quad T_2(\underline{X})\right) = \left(\min_{1\leq i\leq n} X_i, \max_{1\leq i\leq n} X_i\right) = (X_{(1)}, X_{(n)})$$

is sufficient statistics for $\theta = (\theta_1, \theta_2)$.

6.

(Problem 1.5.5)

Let $\theta = (\theta_1, \theta_2)$ be a bivariate parameter. Suppose that $T_1(X)$ is sufficient for θ_1 whenever θ_2 is fixed and known, whereas $T_2(X)$ is sufficient for θ_2 whenever θ_1 is fixed and known.

Assume that θ_1, θ_2 vary independently, $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$ and that the set $S = \{x : p(x,\theta) > 0\}$ does not depend on θ.

(a) Show that if T_1 and T_2 do not depend on θ_2 and θ_1 respectively, then $(T_1(X), T_2(X))$ is sufficient for θ.

(b) Exhibit an example in which $(T_1(X), T_2(X))$ is sufficient for θ, $T_1(X)$ is sufficient for θ_1 whenever θ_2 is fixed and known, but $T_2(X)$ is not sufficient for θ_2, when θ_1 is fixed and known.

<u>Solution:</u>

(a)

Since $T_1(X)$ is sufficient for θ_1 whenever θ_2 is fixed, there should exist

$$
\begin{aligned}
& f(X \mid \theta_1, \theta_2) \\
&= g_1(T_1(X), \theta_1, \theta_2) h_0(X, \theta_2)
\end{aligned}
$$

And also since $T_2(X)$ is sufficient for θ_2 whenever θ_1 is fixed, then

$$h_0(X, \theta_2) = h_1(T_2(X), \theta_2) h(X)$$

And also since $T_1(X)$ does not depend on θ_2, then

$$
\begin{aligned}
& g_1(T_1(X), \theta_1, \theta_2) \\
&= \tilde{g}(T_1(X), \theta_1) \tilde{h}(\theta_1, \theta_2)
\end{aligned}
$$

So

$$
\begin{aligned}
& f(X \mid \theta_1, \theta_2) \\
&= \tilde{\tilde{g}}(T_1(X), T_2(X), \theta_1, \theta_2) h(X)
\end{aligned}
$$

, where

$$
\begin{aligned}
& \tilde{\tilde{g}}(T_1(X), T_2(X), \theta_1, \theta_2) \\
&= \tilde{g}(T_1(X), \theta_1) \tilde{h}(\theta_1, \theta_2) h_1(T_2(X), \theta_2)
\end{aligned}
$$

Then, by factorization theorem, $T(X) = (T_1(X), T_2(X))$ is sufficient statistics for $\theta = (\theta_1, \theta_2)$.

(b) See bivariate normal distribution

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), -\infty < \mu < \infty, \sigma > 0 .$$

It is not difficult to prove that $T(X) = (T_1(X), T_2(X)) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is sufficient statistics for (μ, σ^2).

Further, when σ^2 is fixed, $T_1(X) = \sum_{i=1}^n X_i$ is sufficient statistics for μ.

But when μ is fixed, $T_2(X) = \sum_{i=1}^n X_i^2$ is not sufficient statistics for σ^2.