Stat 543 HW #7, Due Friday 3/24/05

Fisher Information, Cramér-Rao Inequality, Intro to Testing, Neyman-Pearson Lemma

1. Problems 3.3.15, 3.4.11, 3.4.12, 3.4.22 of Bickel and Doksum.

2. Suppose that $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$.
   a) Consider the two-dimensional parameter, $\boldsymbol{\theta} = \left(\mu, \sigma\right)$. Find the Fisher information matrix $I_X\left(\boldsymbol{\theta}_0\right)$.
   b) Then consider the reparameterization in exponential family form with

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\mu}{\sigma^2} \\ -\dfrac{1}{2\sigma^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\theta_1}{\theta_2^2} \\ -\dfrac{1}{2\theta_2^2} \end{pmatrix}$$

   What is $I_X\left(\boldsymbol{\eta}_0\right)$?
   c) If

$$g\left(\boldsymbol{\theta}\right) = \begin{pmatrix} \dfrac{\theta_1}{\theta_2^2} \\ -\dfrac{1}{2\theta_2^2} \end{pmatrix}$$

   and $\boldsymbol{\theta}_0 = g^{-1}\left(\boldsymbol{\eta}_0\right)$, can one simply plug $\boldsymbol{\theta}_0$ into matrix from a) to get the matrix from b)? The complete story hinted at here is told in Problem 3.4.3 of B&D.

3. What is the "Jeffery's prior" for a model with $X \sim \text{Poisson}\left(\lambda\right)$? For this "prior," what is the posterior distribution of $\lambda \mid X$?

4. (Optional only, highly recommended but not required) Suppose that X is a discrete random variable taking values in some finite set $\mathcal{X}$ and that $f\left(x \mid \theta\right) > 0$ for all x for $\theta \in \Theta$ (some open interval in $\Re$). Suppose further for each $x \in \mathcal{X}$ that $f\left(x \mid \theta\right)$ is differentiable in θ at θ_0. Show that for any statistic $T\left(X\right)$,

$$I_{T(X)}\left(\theta_0\right) \le I_X\left(\theta_0\right)$$

(As I recall, this boils down to an appropriate use of Jensen's inequality, considering the various conditional distributions of $X \mid T\left(X\right) = t$.)

5. (Optional only, recommended but not required) Problems 3.4.3, 3.4.20 of B&D (note that in 3.4.20, the "P" must refer to the joint distribution of $\left(X, \theta\right)$ specified by the likelihood and prior)

6. Problems 4.1.1, 4.1.3, 4.2.1, 4.2.3 of B&D

7. (Optional only, recommended but not required) Problems 4.1.5, 4.1.6, 4.2.4, 4.2.5 of B&D