Stat 543 Assignment 10 (Due Friday April 22, 2005)
Asymptotics of "Maximum Likelihood"

1. (Estimation in a Zero-Inflated Poisson Model) Consider n iid discrete observations $X_1, X_2, ..., X_n$, each with marginal probability mass function on $\{0, 1, 2, ...\}$

$$f(x|p,\lambda) = \begin{cases} p\exp(-\lambda) + (1-p) & x = 0 \\ p\frac{\exp(-\lambda)\lambda^x}{x!} & x = 1, 2, 3, ... \end{cases}$$

where $p \in (0,1)$ and $\lambda > 0$. (This marginal is a mixture of a distribution degenerate at 0 and the Poisson (λ) distribution. It might arise in the inspection for flaws of a mixed lot of items, some of which come from a "perfect" process and others of which come from a process that puts flaws on the items according to a Poisson distribution.)

   (a) Give the likelihood equations for this problem.

   (b) What are the (marginal) large sample distributions of method of moments estimators of p and λ , say $\tilde{p}$ and $\tilde{\lambda}$, when $p = .7$ and $\lambda = 3.0$? (Hint: What is the covariance matrix for the vector $(X_1, X_1^2)'$? What, then, does the multivariate central limit theorem say about the large sample joint distribution of $\sqrt{n}\left(\frac{1}{n}\sum X_i - \mathrm{E}_{p,\lambda}X_1, \frac{1}{n}\sum X_i^2 - \mathrm{E}_{p,\lambda}X_1^2\right)'$? What, then, does the delta method give you for the large sample joint distribution of $\sqrt{n}\left(\tilde{p} - .7, \tilde{\lambda} - 3\right)'$?) It may well be useful to know that the first 4 moments of the Poisson distribution are: $\mu_1 = \lambda$, $\mu_2 = \lambda^2 + \lambda$, $\mu_3 = \lambda^3 + 3\lambda^2 + \lambda$ and $\mu_4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$.

   (c) What is the large sample joint distribution of an "MLE" of (p, λ) if $p = .7$ and $\lambda = 3.0$? How do the marginal distributions compare to those in b)?

   Below are $n = 20$ observations that I simulated from this distribution using $p = .7$ and $\lambda = 3.0$.

$$0, 3, 2, 5, 3, 4, 0, 0, 4, 0, 0, 5, 3, 5, 5, 4, 2, 0, 1, 2$$

   (d) Find method of moments estimators based on the data above. Then compute a "one-step Newton improvement" on $(\tilde{p}, \tilde{\lambda})$.

   (e) The MLE of (p, λ) based on these data turns out to be $(\widehat{p}, \widehat{\lambda}) = (.72675, 3.30242)$. Find (individual) large sample 90% confidence intervals for p and λ based on $(\widehat{p}, \widehat{\lambda})$ and the observed Fisher information matrix.

   (f) Find an elliptical large sample 90% joint confidence region for (p, λ) based on $(\widehat{p}, \widehat{\lambda})$ and the observed Fisher information matrix. Plot this in the (p, λ)-plane.

2. Suppose that for $\alpha \in [0,1]$, X is a random variable with probability density

$$f(x|\alpha) = \alpha f_1(x) + (1-\alpha) f_0(x)$$

where $f_0(x)$ is the N$(0,1)$ density and $f_1(x)$ is the N$(1,1)$ density.

   (a) Find the mean and variance of X, $\mathrm{E}_\alpha X$ and $\mathrm{Var}_\alpha X$.

(b) Show that the maximum likelihood estimator of α based on the single observation, X, is

$$\widehat{\alpha} = \begin{cases} 1 & \text{if } X > .5 \\ 0 & \text{otherwise} \end{cases}$$

Compute the mean and variance of this estimator. Is $\widehat{\alpha}$ unbiased for α?

(c) Argue that the mean squared error of $\widehat{\alpha}$ as an estimator of α is no more than $.25 + (.3085)^2$. Use this fact and compare $\widehat{\alpha}$ and X in terms of mean squared error.

(d) Set up an integral giving $I(\alpha)$, the Fisher information in X concerning $\alpha \in (0, 1)$.

Now consider estimation of α based on a sample $X_1, X_2, ..., X_n$ that are iid with density $(*)$. Let $\widehat{\alpha}_n$ be the MLE of α based on the n observations and let $\overline{X}_n$ be the usual sample mean.

(e) The following figure gives plots of both $1/I(\alpha)$ and $1 + \alpha - \alpha^2$. What does this figure indicate about the the large sample distributions of $\widehat{\alpha}_n$ and $\overline{X}_n$? On the basis of large sample considerations, which of these is the better estimator of α? Explain carefully.

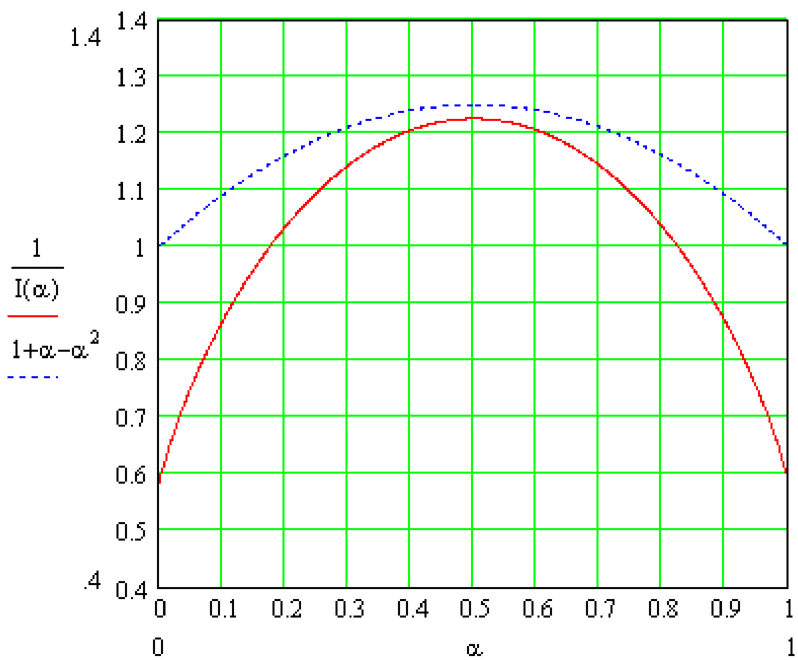

Figure 1: $1/I(\alpha)$ and $1 + \alpha + \alpha^2$

(f) A particular sample of $n = 20$ observations produces $\widehat{\alpha}_n = .4$. What is an approximate 90% confidence interval for α based on the "expected Fisher information" in a single observation? Explain where you are getting your limits.