STAT542 HW9 SOLUTION

4.39

Without loss of generality lets assume that $i < j$. From the discussion in the text we have

$$\begin{aligned}
f(x_1, ..., x_n) &= \frac{m!}{x_1!...x_n!} p_1^{x_1} ... p_n^{x_n} = m! \prod_{i=1}^{n} \frac{p_i^{x_i}}{x_i!} \\
f(x_j) &= \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j} \sum_{x_k \neq x_j} \frac{(m-x_j)!}{x_1!...x_{j-1}!x_{j+1}!...x_n!} \\
&\quad \times \left(\frac{p_1}{1-p_j}\right)^{x_1} ... \left(\frac{p_{j-1}}{1-p_j}\right)^{x_{j-1}} \left(\frac{p_{j+1}}{1-p_j}\right)^{x_{j+1}} ... \left(\frac{p_n}{1-p_j}\right)^{x_n} \\
&= \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j}
\end{aligned}$$

$$\begin{aligned}
&f(x_1, ..., x_n | x_j) = \frac{f(x_1, ..., x_n)}{f(x_j)} \\
&= \frac{(m-x_j)!}{x_1!...x_{j-1}!x_{j+1}!...x_n!} \left(\frac{p_1}{1-p_j}\right)^{x_1} ... \left(\frac{p_{j-1}}{1-p_j}\right)^{x_{j-1}} \left(\frac{p_{j+1}}{1-p_j}\right)^{x_{j+1}} ... \left(\frac{p_n}{1-p_j}\right)^{x_n}
\end{aligned}$$

$$\begin{aligned}
f(x_i | x_j) &= \sum_{x_k \neq x_i, x_j} f(x_1, ..., x_{j-1}, x_{j+1} ... x_n | x_j) \\
&= \sum_{x_k \neq x_i, x_j} \frac{(m-x_j)!}{x_1!...x_{j-1}!x_{j+1}!...x_n!} \\
&\quad \times \left(\frac{p_1}{1-p_j}\right)^{x_1} ... \left(\frac{p_{j-1}}{1-p_j}\right)^{x_{j-1}} \left(\frac{p_{j+1}}{1-p_j}\right)^{x_{j+1}} ... \left(\frac{p_n}{1-p_j}\right)^{x_n} \\
&\quad \times \frac{(m-x_i-x_j)!(1-\frac{p_i}{1-p_j})^{m-x_i-x_j}}{(m-x_i-x_j)!(1-\frac{p_i}{1-p_j})^{m-x_i-x_j}} \\
&= \frac{(m-x_j)!}{x_i!(m-x_i-x_j)!} \left(\frac{p_i}{1-p_j}\right)^{x_i} \left(1-\frac{p_i}{1-p_j}\right)^{m-x_i-x_j} \\
&\quad \times \sum_{x_k \neq x_i, x_j} \frac{(m-x_i-x_j)!}{x_1!...x_{i-1}!x_{i+1}!...x_{j-1}!x_{j+1}!...x_n!} \\
&\quad \times \left(\frac{p_1}{1-p_j-p_i}\right)^{x_1} ... \left(\frac{p_{i-1}}{1-p_j-p_i}\right)^{x_{i-1}} \left(\frac{p_{i+1}}{1-p_j-p_i}\right)^{x_{i+1}} ... \\
&\quad ... \left(\frac{p_{j-1}}{1-p_j-p_i}\right)^{x_{j-1}} \left(\frac{p_{j+1}}{1-p_j-p_i}\right)^{x_{j+1}} ... \left(\frac{p_n}{1-p_j-p_i}\right)^{x_n} \\
&= \frac{(m-x_j)!}{x_i!(m-x_i-x_j)!} \left(\frac{p_i}{1-p_j}\right)^{x_i} \left(1-\frac{p_i}{1-p_j}\right)^{m-x_i-x_j}
\end{aligned}$$

Thus,$X_i|X_j \sim binomial(m - x_j, \frac{p_i}{1-p_j})$. Meanwhile,

$$f(x_i, x_j) = f(x_i|x_j)f(x_j) = \frac{m!}{x_i!x_j!(m - x_i - x_j)!} p_i^{x_i} p_j^{x_j} (1 - p_j - p_i)^{m - x_i - x_j}.$$

Using this result it can be shown that $X_i + X_j \sim binomial(m, p_i + p_j)$. Therefore,

$$\begin{aligned}
Var(X_i + X_j) &= m(p_i + p_j)(1 - p_i - p_j). \\
Cov(X_i, X_j) &= \frac{1}{2}(Var(X_i + X_j) - Var(X_i) - Var(X_j)) \\
&= \frac{1}{2}(m(p_i + p_j)(1 - p_i - p_j) - mp_i(1 - p_i) - mp_j(1 - p_j)) = -mp_ip_j.
\end{aligned}$$

$$\begin{aligned}
E(X_1X_2X_3) &= E(E(X_1X_2X_3|X_3)) \\
&= E(X_3E(X_1X_2|X_3)) \\
&= E(X_3E(E(X_1X_2|X_2, X_3)|X_3)) \\
&= E(X_3E(X_2E(X_1|X_2, X_3)|X_3)) \\
&= E(X_3E(X_2(m - X_2 - X_3)\frac{p_1}{1 - p_2 - p_3}|X_3)) \\
&= \frac{p_1}{1 - p_2 - p_3} E(X_3((mEX_2 - EX_2^2 - EX_2X_3)|X_3)) \\
&= \frac{p_1}{1 - p_2 - p_3} E(X_3(mE(X_2|X_3) - E(X_2^2|X_3) - E(X_2X_3|X_3))) \\
&= \frac{p_1}{1 - p_2 - p_3} E(X_3(m(m - X_3)\left(\frac{p_2}{1 - p_3}\right) - (m - X_3)^2\left(\frac{p_2}{1 - p_3}\right)^2 \\
&\quad -(m - X_3)\left(\frac{p_2}{1 - p_3}\right)\left(\frac{1 - p_2 - p_3}{1 - p_3}\right) - (m - X_3)\left(\frac{p_2}{1 - p_3}\right)X_3)) \\
&= \frac{p_1p_2}{(1 - p_3)^2} E(X_3(m - X_3)(m - X_3 - 1)) \\
&= \frac{p_1p_2}{(1 - p_3)^2} E(X_3^3 - (2m - 1)EX_3^2 + m(m - 1)EX_3) \\
&= \frac{p_1p_2}{(1 - p_3)^2} (m(m - 1)(m - 2)p_3(1 - p_3)^2) \\
&= m(m - 1)(m - 2)p_1p_2p_3
\end{aligned}$$

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Note that given bivariate normal distribution, $X \sim N(0,1)$ and $Y|X = x \sim N(\rho x, 1-\rho^2)$.

$$\begin{aligned}
Cor(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\
&= Cov(X,Y) \\
&= EXY - EXEY \\
&= E(E(XY|X)) - 0 \\
&= E(X|E(Y|X)) \\
&= E(X\rho X) \\
&= \rho E(X^2) \\
&= \rho
\end{aligned}$$

$$\begin{aligned}
Cor(X^2,Y^2) &= \frac{Cov(X^2,Y^2)}{\sqrt{Var(X^2)Var(Y^2)}} \\
&= \frac{E(X^2Y^2) - E(X^2)E(Y^2)}{\sqrt{(EX^4 - (EX^2)^2)(EY^4 - (EY^2)^2)}} \\
&= \frac{E(X^2E(Y^2|X)) - 1}{\sqrt{(3-1^2)(3-1^2)}}, \quad E(X^4) = E(Y^4) = 3 \\
&= \frac{E(X^2(Var(Y|X) + E^2(Y|X))) - 1}{2} \\
&= \frac{E(X^2(1-\rho^2+\rho^2X^2)) - 1}{2} \\
&= \frac{E(X^2) - \rho^2E(X^2) + \rho^2E(X^4) - 1}{2} \\
&= \frac{1-\rho^2+3\rho^2-1}{2} \\
&= \rho^2
\end{aligned}$$

Additional Problems 1.

$$\begin{pmatrix} U \\ V \end{pmatrix} = BX + d, \text{where } B = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$E\begin{pmatrix} U \\ V \end{pmatrix} = B\mu + d = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned}
Var\begin{pmatrix} U \\ V \end{pmatrix} &= B\Sigma B' \\
&= \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 10 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 16 & 1 \\ 1 & 32 \end{pmatrix}
\end{aligned}$$

Thus, $\begin{pmatrix} U \\ V \end{pmatrix} \sim MVN_2\left(\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 16 & 1 \\ 1 & 32 \end{pmatrix}\right)$.

2.

a.

$$\begin{aligned}
M_{\begin{pmatrix} X \\ Y \end{pmatrix}}(t_1, t_2) &= E(e^{t_1 X + t_2 Y}) \\
&= E(E(e^{t_1 X + t_2 Y}|X)) \\
&= E(e^{t_1 X} E(e^{t_2 Y}|X)) \\
&= E(e^{t_1 X} e^{t_2^2 + t_2 X}) \\
&= e^{t_2^2} E(e^{t_1 X + t_2 X}) \\
&= e^{t_2^2} e^{\frac{(t_1+t_2)^2}{2}} \\
&= exp\{\frac{1}{2} t' \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} t\},
\end{aligned}$$

which is the mgf of $\begin{pmatrix} X \\ Y \end{pmatrix} \sim MVN_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\right)$.

b. Given $\begin{pmatrix} X \\ Y \end{pmatrix} \sim MVN_2\{\mu, \Sigma\} = MVN_2\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\right\}$,
then $\Sigma^{-1} = \begin{pmatrix} 2/3 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

$$\frac{3}{2}X^2 - XY + \frac{1}{2}Y^2 = X\Sigma^{-1}X' \sim \chi_2^2.$$

c.

$$\begin{pmatrix} X - Y \\ X + Y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{aligned}
Var\begin{pmatrix} X - Y \\ X + Y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}, \quad \text{which is nonsingular.}
\end{aligned}$$

3.

Let $\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 10$, similar to the results of 4.40,
Marginal $X_i \sim beta(\alpha_i, \alpha_0 - \alpha_i), i = 1, 2, 3, 4$.

$$\begin{aligned}
EX_1 &= \frac{\alpha_1}{\alpha_0} = \frac{1}{10}, \\
EX_2 &= \frac{\alpha_2}{\alpha_0} = \frac{2}{5}, \\
EX_3 &= \frac{\alpha_2}{\alpha_0} = \frac{1}{5}, \\
EX_4 &= \frac{\alpha_4}{\alpha_0} = \frac{3}{10},
\end{aligned}$$

Hence the mean vector is $\left(\frac{1}{10}, \frac{2}{5}, \frac{1}{5}, \frac{3}{10}\right)'$.

$$\begin{aligned}
EX_1^2 &= VarX_1 + (EX_1)^2 = \frac{\alpha_1(\alpha_0 - \alpha_1)}{\alpha_0^2(\alpha_0 + 1)} + \frac{\alpha_1^2}{\alpha_0^2} = \frac{\alpha_1(1 + \alpha_1)}{\alpha_0(1 + \alpha_0)}, \\
EX_1X_2 &= E[E(X_1X_2|X_1)] = E[X_1E(X_2|X_1)] = E\left[X_1(1 - X_1)\frac{\alpha_2}{\alpha_0 - \alpha_1}\right] \\
&= \frac{\alpha_2}{\alpha_0 - \alpha_1}E(X_1 - X_1^2) = \frac{\alpha_2}{\alpha_0 - \alpha_1}\left[\frac{\alpha_1}{\alpha_0 - \alpha_1} - \frac{\alpha_1(\alpha_1 + 1)}{\alpha_0(\alpha_0 + 1)}\right] = \frac{\alpha_1\alpha_2}{\alpha_0(1 + \alpha_1)}, \\
Cov(X_1, X_2) &= EX_1X_2 - EX_1EX_2 = \frac{\alpha_1\alpha_2}{\alpha_0(1 + \alpha_0)} - \frac{\alpha_1}{\alpha_0}\frac{\alpha_2}{\alpha_0} = -\frac{\alpha_1\alpha_2}{\alpha_0^2(1 + \alpha_0)} = -\frac{4}{1100}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
Cov(X_1, X_3) &= -\frac{\alpha_1\alpha_3}{\alpha_0^2(1 + \alpha_0)} = -\frac{2}{1100}, \\
Cov(X_1, X_4) &= -\frac{\alpha_1\alpha_4}{\alpha_0^2(1 + \alpha_0)} = -\frac{3}{1100}, \\
Cov(X_2, X_3) &= -\frac{\alpha_2\alpha_3}{\alpha_0^2(1 + \alpha_0)} = -\frac{8}{1100}, \\
Cov(X_2, X_4) &= -\frac{\alpha_2\alpha_4}{\alpha_0^2(1 + \alpha_0)} = -\frac{12}{1100}, \\
Cov(X_3, X_4) &= -\frac{\alpha_3\alpha_4}{\alpha_0^2(1 + \alpha_0)} = -\frac{6}{1100}, \\
VarX_1 &= \frac{\alpha_1(\alpha_0 - \alpha_1)}{\alpha_0^2(\alpha_0 + 1)} = \frac{9}{1100}, \\
VarX_2 &= \frac{\alpha_2(\alpha_0 - \alpha_2)}{\alpha_0^2(\alpha_0 + 1)} = \frac{24}{1100}, \\
VarX_3 &= \frac{\alpha_3(\alpha_0 - \alpha_3)}{\alpha_0^2(\alpha_0 + 1)} = \frac{16}{1100}, \\
VarX_4 &= \frac{\alpha_4(\alpha_0 - \alpha_4)}{\alpha_0^2(\alpha_0 + 1)} = \frac{21}{1100}.
\end{aligned}$$

Hence covariance matrix is $\frac{1}{1100}\times\begin{pmatrix} 9 & -4 & -2 & -3 \\ -4 & 24 & -8 & -12 \\ -2 & -8 & 16 & -6 \\ -3 & -12 & -6 & 21 \end{pmatrix}$.It is obvious that the covariance matrix is not singular because the sum of 4 variables $x_1+x_2+x_3+x_4$ must equal to 1,ie.,they are linear correlated.

Since X_1, X_2, X_3 and X_4 are Dirichlet distribution, we can write $X_i = \frac{Y_i}{Y_1+Y_2+Y_3+Y_4}$ for $i = 1,2,3,4$, where Y_1, Y_2, Y_3 and X_4 are independent and $Y_i \sim Gamma(\alpha_i, 1)$.

Let $Z = X_1 + X_3 = \frac{Y_1+Y_3}{Y_1+Y_2+Y_3+Y_4}$. Then $Z \sim beta(\alpha_1+\alpha_3, \alpha_2+\alpha_4) = \beta(3,7)$. Thus

$$P[x_1+x_3 > 0.5] = P[Z > 0.5] = \int_{0.5}^{1} \frac{1}{\beta(3,7)} z^{3-1}(1-z)^{7-1}dz = 0.08984$$

4.

For multiple hypergeometric distribution:

$$\begin{aligned}
EX_1 &= n\frac{N_1}{N} = 10\times\frac{10}{100} = 1, \\
EX_2 &= n\frac{N_2}{N} = 40\times\frac{40}{100} = 4, \\
EX_3 &= n\frac{N_3}{N} = 20\times\frac{10}{100} = 2, \\
EX_2 &= n\frac{N_2}{N} = 30\times\frac{40}{100} = 3,
\end{aligned}$$

Hence the mean vector is $(1,4,2,3)'$.

$$\begin{aligned}
VarX_1 &= n\left(\frac{N_1}{N}\right)\left(1-\frac{N_1}{N}\right)\left(\frac{N-n}{N-1}\right) = 10\times\frac{1}{10}\times 910\times 9099 = \frac{9}{11}, \\
VarX_2 &= n\left(\frac{N_2}{N}\right)\left(1-\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right) = 10\times\frac{4}{10}\times 610\times 9099 = \frac{11}{24}, \\
VarX_3 &= n\left(\frac{N_3}{N}\right)\left(1-\frac{N_3}{N}\right)\left(\frac{N-n}{N-1}\right) = 10\times\frac{2}{10}\times 810\times 9099 = \frac{16}{11}, \\
VarX_4 &= n\left(\frac{N_4}{N}\right)\left(1-\frac{N_4}{N}\right)\left(\frac{N-n}{N-1}\right) = 10\times\frac{3}{10}\times 710\times 9099 = \frac{21}{11}, \\
EX_1^2 &= VarX_1 + (EX_1)^2 = n\frac{N_1}{N}\left(1-\frac{N_1}{N}\right)\left(\frac{N-n}{N-1}\right) + n^2\frac{N_1^2}{N^2} \\
&= n\frac{N_1}{N}\left[\left(1-\frac{N_1}{N}\right)\left(\frac{N-n}{N-1}\right) + n\frac{N_1}{N}\right], \\
EX_1X_2 &= E[E(X_1X_2|X_1)] = E\left[X_1(n-X_1)\frac{N_2}{N-N_1}\right] = \frac{N_2}{N-N_1}E(nX_1 - X_1^2) \\
&= \frac{N_2}{N-N_1}\left(n^2\frac{N_1}{N} - n\frac{N_1}{N}\left[\left(1-\frac{N_1}{N}\right)\left(\frac{N-n}{N-1}\right) + n\frac{N_1}{N}\right]\right) = \frac{n(n-1)N_1N_2}{N(N-1)},
\end{aligned}$$

$$\begin{aligned}
Cov(X_1, X_2) &= EX_1X_2 - EX_1EX_2 = \frac{n(n-1)N_1N_2}{N(N-1)} - n\frac{N_1}{N}n\frac{N_2}{N}, \\
&= -\frac{nN_1N_2(N-n)}{N^2(N-1)} = -\frac{4}{11},
\end{aligned}$$

Similarly,

$$\begin{aligned}
Cov(X_1, X_3) &= -\frac{nN_1N_3(N-n)}{N^2(N-1)} = -\frac{2}{11}, \\
Cov(X_1, X_4) &= -\frac{nN_1N_4(N-n)}{N^2(N-1)} = -\frac{3}{11}, \\
Cov(X_2, X_3) &= -\frac{nN_2N_3(N-n)}{N^2(N-1)} = -\frac{8}{11}, \\
Cov(X_2, X_4) &= -\frac{nN_2N_4(N-n)}{N^2(N-1)} = -\frac{12}{11}, \\
Cov(X_3, X_4) &= -\frac{nN_3N_4(N-n)}{N^2(N-1)} = -\frac{6}{11}.
\end{aligned}$$

Hence covariance matrix is $\frac{1}{11}\begin{pmatrix} 9 & -4 & -2 & -3 \\ -4 & 24 & -8 & -12 \\ -2 & -8 & 16 & -6 \\ -3 & -12 & -6 & 21 \end{pmatrix}$.

For multinomial distribution:

$$EX_1 = np_1 = 1, \quad EX_2 = np_2 = 4, \quad EX_3 = np_3 = 2, \quad EX_4 = np_4 = 3$$

Hence the mean vector is (1,4,2,3)'.

$$\begin{aligned}
VarX_1 &= np_1(1-p_1) = 0.9, \\
VarX_2 &= np_2(1-p_2) = 2.4, \\
VarX_3 &= np_3(1-p_3) = 1.6, \\
VarX_4 &= np_4(1-p_4) = 2.1, \\
EX_1X_2 &= E(E(X_1X_2|X_1)) = E(X_1E(X_2|X_1)) = E\left(X_1(n-X_1)\frac{p_2}{1-p_1}\right), \\
&= \frac{p_2}{1-p_1}E(nX_1 - X_1^2), \\
EX_1^2 &= VarX_1 + (EX_1)^2 = np_1(1-p_1) + n^2p_1^2, \\
EX_1X_2 &= \frac{p_2}{1-p_1}[n^2p_1 - np_1(1-p_1) - n^2p_1^2] = n(n-1)p_1p_2, \\
Cov(X_1, X_2) &= EX_1X_2 - EX_1EX_2 = n(n-1)p_1p_2 - n^2p_1p_2 = -np_1p_2 = -0.4.
\end{aligned}$$

Similarly,

$$\begin{aligned}
Cov(X_1, X_3) &= -np_1p_3 = -0.2, \\
Cov(X_1, X_4) &= -np_1p_4 = -0.3, \\
Cov(X_2, X_4) &= -np_2p_4 = -0.8,
\end{aligned}$$

$$\begin{aligned}
Cov(X_2, X_4) &= -np_2p_4 = -1.2, \\
Cov(X_3, X_4) &= -np_3p_4 = -0.6,
\end{aligned}$$

Hence covariance matrix is $\frac{1}{10}\begin{pmatrix} 9 & -4 & -2 & -3 \\ -4 & 24 & -8 & -12 \\ -2 & -8 & 16 & -6 \\ -3 & -12 & -6 & 21 \end{pmatrix}$.

For multivariate normal distribution:
Let Σ be the covariance matrix of multiple hypergeometric distribution,
$MVN4 \sim \left(\begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \end{pmatrix}, \Sigma\right)$ have the same mean vector and covariance matrix as those of multiple hypergeometric distribution.

For dirichlet distribution:
Let X have Dirichlet distribution with parameter $\alpha = (1, 4, 2, 3)$. In question 3 above, we got mean vector and covariance matrix for X. Let $Y = 10X$. We can find that Y has Dirichlet distribution with the same mean vector and covariance matrix as those of the multiple hypergeometric distribution.