STAT542 HW8 SOLUTION

4.16

a. The support of the distribution of (U,V) is $\{(u,v) : u = 1, 2, ...; v = 0, \pm1, \pm2, ...\}$.
   If $V > 0$, then $X > Y$. So for $v = 1, 2, ...$,the joint pmf is

$$\begin{aligned} f_{U,V}(u,v) &= P(U = u, V = v) \\ &= P(Y = u, X = u + v) \\ &= p(1-p)^{u+v-1}p(1-p)^{u-1} \\ &= p^2(1-p)^{2u+v-2}. \end{aligned}$$

   If $V < 0$, then $X < Y$. So for $v = -1, -2, ...$,the joint pmf is

$$\begin{aligned} f_{U,V}(u,v) &= P(U = u, V = v) \\ &= P(X = u, Y = u - v) \\ &= p(1-p)^{u-1}p(1-p)^{u-v-1} \\ &= p^2(1-p)^{2u-v-2}. \end{aligned}$$

   If $V = 0$,then $X = Y$. So for v=0, the joint pmf is

$$\begin{aligned} f_{U,V}(u,v) &= P(U = u, V = 0) \\ &= P(X = u, Y = u) \\ &= p(1-p)^{u-1}p(1-p)^{u-1} \\ &= p^2(1-p)^{2u-2}. \end{aligned}$$

   In all three cases, we can write the joint pmf as

$$\begin{aligned} f_{U,V}(u,v) &= p^2(1-p)^{2u+|v|-2} \\ &= p^2(1-p)^{2u}(1-p)^{|v|-2}. \end{aligned}$$

   Since the joint pmf factors into a function of u and a function of v, U and V are independent.

b. skip

c.

$$\begin{aligned} P(X = x, X + Y = t) &= P(X = x, Y = t - x) \\ &= P(X = x)P(Y = t - x) \\ &= p^2(1-p)^{t-2}. \end{aligned}$$

4.23

a. Let $y = v, x = u/y = u/v$ then

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} (\frac{u}{v})^{\alpha-1}(1-\frac{u}{v})^{\beta-1} v^{\alpha+\beta-1}(1-v)^{\gamma-1}\frac{1}{v}, \quad 0 < u < v < 1.$$

$$\begin{aligned} f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 v^{\beta-1}(1-v)^{\gamma-1}(\frac{v-u}{v})^{\beta-1} dv \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_0^1 y^{\beta-1}(1-y)^{\gamma-1} dy, \\ &\quad (\text{let} \quad y = \frac{v-u}{1-u}, dy = \frac{dv}{1-u}) \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \end{aligned}$$

Thus, U is gamma$(\alpha, \beta+\gamma)$.

b. Let $x = \sqrt{uv}, y = \sqrt{\frac{u}{v}}$ then

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}v^{1/2}u^{-1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \\ \frac{1}{2}v^{-1/2}u^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \end{vmatrix} = \frac{1}{2v}$$

$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} (\sqrt{uv})^{\alpha-1}(1-\sqrt{uv})^{\beta-1} \left(\sqrt{\frac{u}{v}}\right)^{\alpha+\beta-1} \left(1-\sqrt{\frac{u}{v}}\right)^{\gamma-1} \frac{1}{2v}$$

The set$\{0 < x < 1, 0 < y < 1\}$ is mapped on to the set$\{0 < u < v < \frac{1}{u}, 0 < u < 1\}$.

$$\begin{aligned} f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \\ &\quad \int_u^{1/u} \left(\frac{1-\sqrt{uv}}{1-u}\right)^{\beta-1} \left(\frac{1-\sqrt{u/v}}{1-u}\right)^{\gamma-1} \frac{(\sqrt{u/v})^{\beta}}{2v(1-u)} dv \\ &\quad (\text{let} \quad z = \frac{\sqrt{u/v}-u}{1-u}, dz = \frac{\sqrt{u/v}dv}{2(1-u)v}) \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_0^1 z^{\beta-1}(1-z)^{\gamma-1} dz \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}, \quad 0 < u < 1. \end{aligned}$$

Thus, U is beta$(\alpha, \beta+\gamma)$.

4.26

a.

$$\begin{aligned} P(Z \le z, W=0) &= P(min(X,Y) \le z, Y \le X) \\ &= P(Y \le z, Y \le X) \\ &= \int_0^z \int_y^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy \\ &= \frac{\lambda}{\mu+\lambda}(1 - e^{(-\frac{1}{\mu}+\frac{1}{\lambda})z}). \end{aligned}$$

Similarly,

$$\begin{aligned} P(Z \le z, W=1) &= P(min(X,Y) \le z, X \le Y) \\ &= P(X \le z, X \le Y) \\ &= \int_0^z \int_x^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy \\ &= \frac{\mu}{\mu+\lambda}(1 - e^{(-\frac{1}{\mu}+\frac{1}{\lambda})z}). \end{aligned}$$

b.

$$\begin{aligned} P(W=0) &= P(Y \le X) \\ &= \int_0^\infty \int_y^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy \\ &= \frac{\lambda}{\mu+\lambda} \end{aligned}$$

$$\begin{aligned} P(W=1) &= P(X \le Y) \\ &= \int_0^\infty \int_x^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy \\ &= \frac{\mu}{\mu+\lambda} \end{aligned}$$

$$P(Z \le z) = P(Z \le z, W=0) + P(Z \le z, W=1) = 1 - e^{(-\frac{1}{\mu}+\frac{1}{\lambda})z}$$

$$P(Z \le z | W=0) = \frac{P(Z \le z, W=0)}{P(W=0)} = 1 - e^{(-\frac{1}{\mu}+\frac{1}{\lambda})z} = P(Z \le z)$$

$$P(Z \le z | W=1) = \frac{P(Z \le z, W=1)}{P(W=1)} = 1 - e^{(-\frac{1}{\mu}+\frac{1}{\lambda})z} = P(Z \le z)$$

Therefore, Z and W are independent.

4.28

a.

$$f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}, \quad -\infty < x < \infty, -\infty < y < \infty.$$

Let $u = \frac{x}{x+y}, v = x+y$, then $x = uv, y = v - uv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v$$

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv, v-uv)|J| \\ &= \frac{1}{2\pi}e^{\frac{v^2}{2}(2u^2-2u+1)}|v|, \quad -\infty < u < \infty, -\infty < v < \infty. \end{aligned}$$

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-\frac{v^2}{2}(2u^2-2u+1)|v|}dv \\ &= \frac{1}{\pi}\int_0^{\infty} e^{-\frac{v^2}{2}(2u^2-2u+1)v}dv \\ &= \frac{1}{\pi}\cdot\frac{1}{2u^2-2u+1} \end{aligned}$$

Hence, $\frac{X}{X+Y}$ is Cauchy$(\frac{1}{2}, \frac{1}{2})$.

b. Let $u = \frac{x}{|y|}, v = |y|$, then $A_1 = \{(x,y) \in R^2 : y > 0\}$ such that $x = uv, y = v$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Meanwhile, $A_2 = \{(x,y) \in R^2 : y < 0\}$ such that $x = uv, y = -v$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & -1 \end{vmatrix} = -v$$

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi}e^{-\frac{1}{2}(u^2v^2+v^2)}|v| + \frac{1}{2\pi}e^{-\frac{1}{2}(u^2v^2+v^2)}|-v| \\ &= \frac{v}{\pi}e^{-\frac{1}{2}(u^2v^2+v^2)}, \quad -\infty < u < \infty, v > 0. \end{aligned}$$

$$\begin{aligned} f_U(u) &= \int_0^{\infty} \frac{v}{\pi}e^{-\frac{1}{2}(u^2v^2+v^2)}dv \\ &= \frac{1}{\pi}\cdot\frac{1}{u^2+1}, \quad -\infty < u < \infty. \end{aligned}$$

4.31

a.

$$\begin{aligned} EY &= E(E(Y|X)) = E(nX) = \frac{n}{2}. \\ VarY &= Var(E(Y|X)) + E(Var(Y|X)) = Var(nX) + E(nX(1-X)) = \frac{n^2}{12} + \frac{n}{6}. \end{aligned}$$

b.

$$P(Y = y, X \le x) = \binom{n}{y} x^y (1-x)^{(n-y)}, \quad y = 0, 1, ..., n, 0 < x < 1.$$

c.

$$\begin{aligned} P(Y = y) &= \int_0^1 \binom{n}{y} x^y (1-x)^{(n-y)} dx \\ &= \binom{n}{y} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \\ &= \frac{1}{n+1}, \quad y = 0, 1, ..., n. \end{aligned}$$

4.32

a.

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_Y(y|\lambda) f_\Lambda(\lambda) d\lambda \\ &= \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{y+\alpha-1} e^{-\lambda(1+\beta)/\beta} d\lambda \\ &= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{1+\beta}\right)^{y+\alpha} \end{aligned}$$

if α is a positive integer,

$$f_Y(y) = \binom{y+\alpha-1}{y} \left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha,$$

which is the pmf of negative binomial $\left(\alpha, \frac{1}{(1+\beta)}\right)$.

$$\mathrm{E}Y = \mathrm{E}(\mathrm{E}(Y|\Lambda)) = \mathrm{E}\Lambda = \alpha\beta$$

$$\mathrm{Var}Y = \mathrm{Var}(\mathrm{E}(Y|\Lambda)) + \mathrm{E}(\mathrm{Var}(Y|\Lambda)) = \mathrm{Var}\Lambda + \mathrm{E}\Lambda = \alpha\beta^2 + \alpha\beta.$$

b.

$$\begin{aligned}
P(Y=y|\lambda) &= \sum_{n=y}^{\infty} P(Y=y|N=n,\lambda)P(N=n|\lambda) \\
&= \sum_{n=y}^{\infty} \binom{n}{y} p^y(1-p)^{n-y}\frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n=y}^{\infty} \frac{1}{y!(n-y)!}\left(\frac{p}{1-p}\right)^y ((1-p)\lambda)^n e^{-\lambda} \\
&= e^{-\lambda}\left(\frac{p}{1-p}\right)^y \sum_{m=0}^{\infty}\frac{1}{y!m!}((1-p)\lambda)^{m+y}, \quad \text{let } m=n-y, \\
&= \frac{e^{-\lambda}}{y!}\left(\frac{p}{1-p}\right)^y ((1-p)\lambda)^y \sum_{m=0}^{\infty}\frac{((1-p)\lambda)^m}{m!} \\
&= \frac{e^{-\lambda}(p\lambda)^y e^{(1-p)\lambda}}{y!} \\
&= \frac{(p\lambda)^y e^{-p\lambda}}{y!}
\end{aligned}$$

Thus $Y|\Lambda \sim$ Poisson$(p\lambda)$, use the use the calculations like those in a) yield the pmf of Y is

$$f_(y) = \frac{1}{\Gamma(\alpha)y!(p\beta)^\alpha}\Gamma(y+\alpha)\left(\frac{p\beta}{1+p\beta}\right)^{y+\alpha}, \quad y=0,1,...,.$$

Again,if α is positive, $Y \sim$negative binomial$\left(\alpha, \frac{1}{1+p\beta}\right)$.

4.47

a. By definition of Z, for $z<0$,

$$\begin{aligned}
P(Z\le z) &= P(X\le z \text{ and } XY>0) + P(-X\le z \quad \text{and} \quad XY<0) \\
&= P(X\le z \quad \text{and} \quad Y<0) + P(-X\le z \quad \text{and} \quad Y<0) \\
&= P(X\le z \quad \text{and} \quad Y<0) + P(X\ge -z \quad \text{and} \quad Y<0) \\
&= P(X\le z)P(Y<0) + P(X\ge -z)P(Y>0) \quad \text{(independence)} \\
&= P(X\le z)P(Y<0) + P(X\le z)P(Y<0) \quad \text{(symmetry of X and Y)} \\
&= P(X\le z)
\end{aligned}$$

By a similar argument, for $z>0$,we can get $P(Z>z)=P(X>z)$. $X\sim n(0,1)$, thus $Z\sim n(0,1)$.

b. By definition of Z, if $Z>0$, then (i)$X>0$ and $Y>0$ or (ii)$X<0$ and $Y>0$. Similarly, if $Z<0$, then (i)$X<0$ and $Y<0$ or (ii)$X>0$ and $Y<0$. So Z always has the same sign as Y. They can not be bivariate normal.

4.55

Let $X = \max(X_1, X_2, X_3)$,

$$P(X \le x) = P(\max(X_1, X_2, X_3) \le x) = P(X_1 \le x)P(X_2 \le x)P(X_3 \le x)$$

Since $P(X_i \le x) = \int_0^x \frac{1}{\lambda} e^{-y/\lambda} dy = 1 - e^{-x/\lambda}$, i=1,2,3,

$$P(X \le x) = \left(1 - e^{-x/\lambda}\right)^3, \qquad 0 < x < \infty.$$

$$f_x(x) = \begin{cases} \frac{3}{\lambda}\left(1 - e^{-x/\lambda}\right)^2 e^{-x/\lambda} & x > 0 \\ 0 & x \le 0 \end{cases}$$