STAT542 HW7 SOLUTION

4.7

We will measure time in minutes past 8 A.M. So X$\sim$uniform(0,30),Y$\sim$uniform(40,50) and the joint pdf is 1/300 on the rectangle $(0,30)\times(40,50)$.

$$P(\text{arrive before 9 A.M.}) = P(X+y<60) = \int_{40}^{50}\int_{0}^{60-y}\frac{1}{300}dxdy = \frac{1}{2}$$

4.10

a. The marginal distribtion of X is $P(X=1)=P(X=3)=1/4$ and $P(X=2)=1/2$. The marginal distribution of Y is $P(Y=2)=P(Y=3)=P(Y=4)=1/3$. But

$$P(X=2,Y=3)=0\neq(\frac{1}{2})(\frac{1}{3})=P(X=2)P(Y=3).$$

Therefore, the random variables are not independent.

b. The distribution that satisfies $P(U=x,V=y)=P(U=x)P(V=y)$ where U$\sim$X and V$\sim$Y is

Distribution of U and V

		U			
		1	2	3	p(V)
	2	1/12	1/6	1/12	1/3
V	3	1/12	1/6	1/12	1/3
	4	1/12	1/6	1/12	1/3
	p(U)	1/4	1/2	1/4	

4.11

The support of the distributionof (U,V)is $\{(u,v): u=1,2,3,...; v=u+1,u+2,u+3,...\}$. This is not a cross-product set. Therefore, U and V are not independent. More simply, if we know $U=u$, then we know $V>u$.

4.22

$$\begin{aligned} F_{U,V}(u,v) &= F_{U,V}(U\leq u, V\leq v) \\ &= F_{U,V}(aX+b\leq u, cY+d\leq v) \\ &= F_{X,Y}(X\leq\frac{u-b}{a}, Y\leq\frac{v-d}{c}) \end{aligned}$$

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\partial F^2_{XY}(\frac{u-b}{a},\frac{v-d}{c})}{\partial u \partial v} \\ &= f_{X,Y}\left(\frac{u-b}{a},\frac{v-d}{c}\right)\cdot|J| \\ &= \frac{1}{ac} f_{X,Y}\left(\frac{u-b}{a},\frac{v-d}{c}\right) \end{aligned}$$

where

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/a & 0 \\ 0 & 1/c \end{vmatrix} = \frac{1}{ac}$$

4.30

a.

$$\begin{aligned} EY &= E[E(Y|X)] = E[X] = \int_0^1 x dx = \frac{1}{2} \\ VarY &= Var[E(Y|X)] + E[Var[Y|X)] = Var(X) + E(X^2) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12} \\ EXY &= E[E(XY|X)] = E(XE(Y|X)) = EX^2 = \frac{1}{3} \\ Cov(X,Y) &= EXY - EXEY = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \end{aligned}$$

b. let $U = Y/X$ and $V = X$. Then $X = V$, $Y = UV$ and $|J| = v$.

$$\begin{aligned} f_{X,Y}(x,y) &= f_{X,Y}(y|x)f_X(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{(y-x)^2}{2x^2}}, 0 < x < 1,\ -\infty < y < \infty \\ f_{U,V}(u,v) &= f_{X,Y}(v,uv)\,|J| \cdot = v \cdot \frac{1}{\sqrt{2\pi}v} e^{-\frac{(uv-v)^2}{2v^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-1)^2}{2}} \end{aligned}$$

Since $f_{U,V}(u,v)$ is free of v, U and V are independent, that is, Y/X and X are independent.The conditional distribution of $U|X = x$ is n(1,1).

4.58

a.

$$\begin{aligned} Cov(X,Y) &= EXY - EXEY \\ &= E(E(XY|X)) - EXE(E(Y|X)) \\ &= E(XE(Y|X)) - EXE(E(Y|X)) \\ &= Cov(X, E(Y|X)) \end{aligned}$$

b.

$$Cov(X, Y - E(Y|X)) = Cov(X,Y) - Cov(X, E(Y|X)) = 0 \quad \text{(by a)}$$

c.

$$\begin{aligned} Var(Y-E(Y|X)) &= E(Var((Y-E(Y|X))|X))+Var(E((Y-E(Y|X))|X)) \\ &= E(Var(Y|X))+Var(E(Y|X)-E(Y|X)) \\ &= E(Var(Y|X)) \end{aligned}$$

4.59

$$\begin{aligned} & E(Cov(X,Y|Z))+Cov(E(X|Z),E(Y|Z)) \\ = & E(E(XY|Z)-E(X|Z)E(Y|Z))+E(E(X|Z)E(Y|Z))-E(E(X|Z))E(E(Y|Z)) \\ = & E(XY)-E(X|Z)E(Y|Z)+E(X|Z)E(Y|Z)-EXEY \\ = & E(XY)-EXEY \\ = & Cov(X,Y) \end{aligned}$$

4.43

$$VarX = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} = \sigma^2 I$$

$$Cov(X1+X2, X2+X3) = (1 \quad 1 \quad 0) VarX \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \sigma^2$$

$$Cov(X1+X2, X1-X2) = (1 \quad 1 \quad 0) VarX \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$