STAT542 HW6 SOLUTION

3.17

$$
\begin{aligned}
EX^\nu &= \int_0^\infty x^\nu \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(\nu+\alpha)-1} e^{-x/\beta} dx \\
&= \frac{\Gamma(\nu+\alpha)\beta^{\nu+\alpha}}{\Gamma(\alpha)\beta^\alpha} = \frac{\beta^\nu \Gamma(\nu+\alpha)}{\Gamma(\alpha)}
\end{aligned}
$$

Note that this formula is valid for all $\nu > -\alpha$. The expectation does not exist for $\nu \leq -\alpha$.

3.20

a.

$$
\begin{aligned}
EX &= \int_0^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} d(\frac{x^2}{2}) \\
&= \frac{2}{\sqrt{2\pi}} (-e^{-\frac{x^2}{2}})|_0^\infty = \frac{2}{\sqrt{2\pi}} [0-(-1)] = \sqrt{\frac{2}{\pi}} \\
EX^2 &= \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 1 \quad (X \sim \text{Normal}(0,1)) \\
VarX &= EX^2 - (EX)^2 = 1 - (\frac{2}{\sqrt{2\pi}})^2 = 1 - \frac{2}{\pi}
\end{aligned}
$$

b. Let $Y = g(X) = X^2$. Thus $X = g^{-1}(Y) = \sqrt{Y}$,where $X, Y \geq 0$.

$$
f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} = \Gamma(1/2, 2)
$$

where $\alpha = 1/2, \beta = 2$.

3.24

a. $f_X(x) = \frac{1}{\beta} e^{-x/\beta}, x > 0$. For $Y = X^{1/\gamma}$,$f_Y(y) = \frac{\gamma}{\beta} e^{-y^\gamma/\beta} y^{\gamma-1}, y > 0$.
$\int_0^\infty f_Y(y) dy = \int_0^\infty \frac{\gamma}{\beta} e^{-y^\gamma/\beta} y^{\gamma-1} dy = -e^{-y^\gamma/\beta}|_0^\infty = 1$.
Using the transformation $z = y^\gamma/\beta$, we calculate

$$
EY^n = \frac{\gamma}{\beta} \int_0^\infty y^{\gamma+n+1} e^{-y^\gamma/\beta} dy = \beta^{n/\gamma} \int_0^\infty z^{n/\gamma} e^{-z} dz = \beta^{n/\gamma} \Gamma(\frac{n}{\gamma} + 1)
$$

Thus $EY = \beta^{1/\gamma} \Gamma(\frac{1}{\gamma} + 1)$ and $VarY = \beta^{2/\gamma} [\Gamma(\frac{2}{\gamma} + 1) - \Gamma^2(\frac{1}{\gamma} + 1)]$.

b. The gamma(a,b) density is $f_X(x) = \frac{1}{\Gamma(a)b^a} x^{a+1} e^{-x/b}$.

Make the transformation $Y = 1/X$ to get $f_Y(y) = \frac{1}{\Gamma(a)b^a} (\frac{1}{y})^{a+1} e^{-\frac{1}{by}}$.

$\int_{-\infty}^{\infty} f_Y(y)dy = \int_0^{\infty} \frac{1}{\Gamma(a)b^a} (\frac{1}{y})^{a+1} e^{-\frac{1}{by}} dy = \int_0^{\infty} \frac{1}{\Gamma(a)b^a} x^{a+1} e^{-x/b} dx = 1.$

The first two moments are

$$\begin{aligned} EY &= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} (\frac{1}{y})^a e^{-\frac{1}{by}} dy = \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} = \frac{a}{(a-1)b} \\ EY^2 &= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} (\frac{1}{y})^{a-1} e^{-\frac{1}{by}} dy = \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} = \frac{1}{(a-1)(a-2)b^2} \\ VarY &= \frac{1}{(a-1)^2(a-2)b^2} \end{aligned}$$

c. $f_X(x) = e^{-x}, x > 0$.

For $Y = \alpha - \gamma log X$, $f_Y(y) = e^{-e^{\frac{\alpha-y}{\gamma}}} e^{\frac{\alpha-y}{\gamma}} \frac{1}{\gamma}, -\infty < y < \infty$.

$\int_{-\infty}^{\infty} f_Y(y)dy = \int_0^{\infty} e^{-e^{\frac{\alpha-y}{\gamma}}} e^{\frac{\alpha-y}{\gamma}} \frac{1}{\gamma} dy = \int_0^{\infty} e^{-x} dx = 1.$

Calculation of EY and EY^2 cannot be done in closed form. If we define

$$\begin{aligned} I_1 &= \int_0^{\infty} log x e^{-x} dx \\ I_2 &= \int_0^{\infty} (log x)^2 e^{-x} dx \end{aligned}$$

then $EY = E(\alpha - \gamma log X) = \alpha - \gamma I_1$, and $EY^2 = E(\alpha - \gamma log X)^2 = \alpha^2 - 2\alpha\gamma I_1 + \gamma^2 I_2$, then $VarY = \gamma^2(I_2 - I_1^2)$. The constant $I_1 = .5772$ is called Euler's constant.

3.26

a. $f_T(t) = \frac{1}{\beta} e^{-t/\beta}, t \geq 0$,and $F_T(t) = \int_0^t \frac{1}{\beta} e^{-x/\beta} dx = -e^{-x/\beta}|_0^t = 1 - e^{-t/\beta}$.Thus,

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{(1/\beta)e^{-t/\beta}}{1 - (1 - e^{-t/\beta})} = \frac{1}{\beta}$$

b. $f_T(t) = \frac{\gamma}{\beta} t^{\gamma-1} e^{-t^\gamma/\beta}, t \geq 0$,

and $F_T(t) = \int_0^t \frac{\gamma}{\beta} x^{\gamma-1} e^{-x^\gamma/\beta} dx = \int_0^{t^\gamma/\beta} e^{-u} du = -e^{-u}|_0^{t^\gamma/\beta} = 1 - e^{-t^\gamma/\beta}$,where $u = x^{\gamma/\beta}$. Thus,

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{(\gamma/\beta)t^{\gamma-1} e^{-t^\gamma/\beta}}{e^{-t^\gamma/\beta}} = \frac{\gamma}{\beta} t^{\gamma-1}.$$

c. $F_T(t) = \frac{1}{1+e^{-(t-\mu)/\beta}}$ and $f_T(t) = \frac{\frac{1}{\beta} e^{-(t-\mu)/\beta}}{(1+e^{-(t-\mu)/\beta})^2}$.

Thus,

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{\frac{1}{\beta} \frac{e^{-(t-\mu)/\beta}}{(1+e^{-(t-\mu)/\beta})^2}}{1 - \frac{1}{1+e^{-(t-\mu)/\beta}}} = \frac{1}{\beta} F_T(t).$$

4.1

Since the distribution is uniform, the easiest way to calculate these probailities is as the ratio of areas, the total areal being 4.

a. The circle $x^2 + y^2 \leq 1$ has area π, so $P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}$.

b. The area below the line y=2x is half of the area of the square, so $P(2X - Y > 0) = \frac{1}{2}$.

c. Clearly $P(|X + Y| < 2) = 1$.

4.5

a. $P(X > \sqrt{Y}) = \int_0^1 \int_{\sqrt{y}}^1 (x + y) dx dy = \frac{7}{20}$

b. $P(X^2 < Y < X) = \int_0^1 \int_y^{\sqrt{y}} 2x dx dy = \frac{1}{6}$

4.40

a.

$$\begin{aligned}
\int\int_{R^2} f(x,y)dxdy &= \int_0^1 \int_0^{1-x} Cx^{a-1}y^{b-1}(1-x-y)^{c-1}dydx \\
&= C\int_0^1 x^{a-1}(1-x)^{b+c-1} \int_0^{1-x} (\frac{y}{1-x})^{b-1}(1-\frac{y}{1-x})^{c-1}d(\frac{y}{1-x})dx \\
&= C\int_0^1 x^{a-1}(1-x)^{b+c-1}B(b,c)dx \\
&= C\,B(b,c)\,B(a,b+c) \equiv 1.
\end{aligned}$$

Hence $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$.

b.

$$\begin{aligned}
\int_R f(x,y)dy &= \int_0^{1-x} Cx^{a-1}y^{b-1}(1-x-y)^{c-1}dy \\
&= Cx^{a-1}\int_0^{1-x} y^{b-1}(1-x-y)^{c-1}dy \\
&= Cx^{a-1}(1-x)^{b+c-1}B(b,c) \\
&= B(a,b+c)x^{a-1}(1-x)^{b+c-1}, 0 \leq x \leq 1.
\end{aligned}$$

Hence X is $Beta(a, b+c)$. Similarly, Y is $Beta(b, a+c)$.

$$\begin{aligned}
\int_R f(x,y)dx &= \int_0^{1-y} Cx^{a-1}y^{b-1}(1-x-y)^{c-1}dx \\
&= Cy^{b-1}\int_0^{1-y} x^{a-1}(1-x-y)^{c-1}dy \\
&= Cy^{b-1}(1-y)^{a+c-1}B(a,c) \\
&= B(b,a+c)y^{b-1}(1-y)^{a+c-1}, 0 \leq y \leq 1.
\end{aligned}$$

c. $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{1}{B(b,c)}(\frac{y}{1-x})^{b-1}(1-\frac{y}{1-x})^{c-1}\frac{1}{1-x}, 0 \le y \le 1$. Let $Z = \frac{Y}{1-x}$,
$f_Z(z) = f_{Y|X=x}((1-x)z)|1-x| = \frac{1}{B(b,c)}z^{b-1}(1-z)^{c-1}, 0 \le z \le 1$.
Hence Z is Beta(b,c).

d.

$$
\begin{aligned}
EXY &= \int\int_{R^2} xyf(x,y)dxdy \\
&= \int_0^1 \int_0^{1-x} xyCx^{a-1}y^{b-1}(1-x-y)^{c-1}dydx \\
&= C\int_0^1 x^a \int_0^{1-x} y^b(1-x-y)^{c-1}dydx \\
&= C\int_0^1 x^a(1-x)^{b+c}B(b+1,c)dx \\
&= CB(b+1,c)B(a+1,b+c+1) \\
&= \frac{ab}{(a+b+c+1)(a+b+c)}. \\
COV(X,Y) &= EXY - EXEY \\
&= \frac{ab}{(a+b+c+1)(a+b+c)} - \frac{a}{a+b+c}\frac{b}{a+b+c} \\
&= -\frac{ab}{(a+b+c)^2(a+b+c+1)}
\end{aligned}
$$

4.43

$$
\begin{aligned}
COV(X_1+X_2, X_2+X_3) &= E(X_1+X_2)(X_2+X_3) - E(X_1+X_2)E(X_2+X_3) \\
&= (4\mu^2+\sigma^2) - 4\mu^2 = \sigma^2 \\
COV(X_1+X_2, X_1-X_2) & \\
&= E(X_1+X_2)(X_1-X_2) - E(X_1+X_2)E(X_1-X_2) \\
&= EX_2^2 - EX_1^2 \\
&= 0
\end{aligned}
$$

Additional Problem

Let X be standard normal N(0,1),

$$
\begin{aligned}
M_X(t) &= \int_{\infty}^{\infty} e^{tX}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-t)^2}{2}+\frac{t^2}{2}}dx \\
&= e^{\frac{t^2}{2}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-t)^2}{2}}dx \\
&= e^{\frac{t^2}{2}}.
\end{aligned}
$$

Let $Y = \mu + \sigma X$, then

$$\begin{aligned} M_Y(t) &= M_{\mu+\sigma X}(t) \\ &= e^{\mu t} M_X(\sigma t) \\ &= e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}}, \end{aligned}$$

which is same as the formula provided in the Casella and Berger.