STAT542 HW5 SOLUTION

3.3

Let X_i = event that a car passes in second i.

$$
\begin{aligned}
& P\{\text{The pedestrian has to wait 4 seconds}\} \\
\Leftrightarrow\ & P\{\text{at least on car in the first 3 seconds}\} \cap P\{\text{a car in the } 4th \text{ second}\} \\
& \quad \cap P\{\text{no car in the last 3 second}\} \\
=\ & P\left((X_1^c \cap X_2^3 \cap X_3^c)^c \cap X_4 \cap X_5^c \cap X_6^c \cap X_7^c\right) \\
=\ & [1 - P(X_1^c \cap X_2^c \cap X_3^c)] \cdot P(X_4) \cdot P(X_5^c) \cdot P(X_6^c) \cdot P(X_7^c) \\
=\ & [1-(1-p)^3]p(1-p)^3
\end{aligned}
$$

3.5

Let X = number of effective cases. If the new and old drugs are equally effective, then the probability that the new drug is effective on a case is .8. If the cases are independent then X is binomial(100, .8), and

$$P(X > 85) = \sum_{k=85}^{100} \binom{100}{k} .8^k .2^{100-k} = .1285$$

So, even if the new drug is no better than the old, the chance of 85 or more effective cases is not too small. Hence, we cannot conclude the new drug is better.

3.7

Let $X \sim$ Poisson(λ). We want $P(X \geq 2) \geq .99$, that is,

$$P(X \leq 1) = e^{-\lambda} + \lambda e^{-\lambda} \leq .01.$$

Solving $e^{-\lambda} + \lambda e^{-\lambda} = .01$ by trial and error yields $\lambda = 6.6384$.

3.8 a.

We want $P(X > N) < .01$ where $X \sim \text{binomial}(1000, 1/2)$. Since the 1000 customers choose randomly, we take $p = 1/2$. We thus require

$$P(X > N) = \sum_{k=N+1}^{1000} \binom{1000}{k} 0.5^k (1-0.5)^{1000-k} < 0.01$$

which implies

$$0.5^{1000} \sum_{k=N+1}^{1000} \binom{1000}{k} < 0.01$$

This last inequality can be used to solve for N, that is, N is the smallest integer that satisfies the above inequality. The solution is $N = 537$.

3.9

a. We can think of each one of the 60 children entering kindergarten as 60 independent Bernoulli trials with probability of success (a twin birth) of approximately $\frac{1}{90}$. The probability of having 5 or more successes approximates the probability of having 5 or more sets of twins entering kindergarten. Then $X \sim \text{binomial}(60, \frac{1}{90})$ and

$$P(x \geq 5) = 1 - \sum_{x=0}^{4} \binom{60}{x} \left(\frac{1}{90}\right)^{x} \left(1 - \frac{1}{90}\right)^{60-x} = 0.0006$$

, which is small and may be rare enough to be newsworthy.

b. Let X be the number of elementary schools in New York state that have 5 or more sets of twins entering kindergarten. Then the probability of interest is $P(X \geq 1)$ where $X \sim \text{binomial}(310, .0006)$. Therefore $P(X \geq 1) = 1 - P(X = 0) = .1698$.

c. Let X be the number of States that have 5 or more sets of twins entering kindergarten during any of the last ten years. Then the probability of interest is $P(X \geq 1)$ where $X \sim \text{binomial}(500, .1698)$. Therefore $P(X \geq 1) = 1 - P(X = 0) = 1 - 3.90 \times 10^{-41} \approx 1$.

3.13

For any X wiht support 0,1,..., we have the mean and variance of the 0-truncated X_t are given by

$$\begin{aligned} EX_T &= \sum_{x=1}^{\infty} xP(X_T = x) \\ &= \sum_{x=1}^{\infty} x\frac{P(X = x)}{P(X > 0)} \\ &= \frac{1}{P(X > 0)} \sum_{x=1}^{\infty} xP(X = x) \\ &= \frac{1}{P(X > 0)} \sum_{x=0}^{\infty} xP(X = x) \\ &= \frac{EX}{P(X > 0)} \end{aligned}$$

In a similar way we get $EX_T^2 = \frac{EX^2}{P(X>0)}$. Thus, $VarX_T = \frac{EX^2}{P(X>0)} - (\frac{EX}{P(X>0)})^2$

a. For Poisson(λ),$P(X > 0) = 1 - P(X = 0) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$,therefore

$$\begin{aligned} P(X_T = x) &= \frac{e^{-\lambda}\lambda^x}{x!(1 - e^{-\lambda})} \\ EX_T &= \lambda/(1 - e^{-\lambda}) \\ VarX_T &= (\lambda^2 + \lambda)/(1 - e^{-\lambda}) - (\lambda/(1 - e^{-\lambda}))^2 \end{aligned}$$