

2.25

- a. Let $Y = g(X) = -X$. Since $g(X)$ is a monotone function, we have $g^{-1}(y) = -y$. And $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = f_X(-y) | -1 | = f_X(y)$ for every y .
- b. To show that $M_X(t)$ is symmetric about 0 we must show that $M_X(0 + \varepsilon) = M_X(0 - \varepsilon)$ for all $\varepsilon > 0$.

$$\begin{aligned} M_X(0 + \varepsilon) &= \int_{-\infty}^{\infty} e^{0+\varepsilon} f_X(x) dx \\ &= - \int_{\infty}^{-\infty} e^{-\varepsilon y} f_X(-y) dy \quad (\text{change variable } x = -y) \\ &= \int_{-\infty}^{\infty} e^{-\varepsilon y} f_X(y) dy \quad (f_X \text{ is even}) \\ &= M_X(0 - \varepsilon) \end{aligned}$$

2.28

a.

$$\begin{aligned} \mu_3 &= \int_{-\infty}^{\infty} (x - a)^3 f(x) dx = \int_{-\infty}^a (x - a)^3 f(x) dx + \int_a^{\infty} (x - a)^3 f(x) dx \\ &= \int_{-\infty}^0 y^3 f(y + a) dy + \int_0^{\infty} y^3 f(y + a) dy \quad (\text{change variable } y = x - a) \\ &= \int_0^{\infty} -y^3 f(-y + a) dy + \int_0^{\infty} y^3 f(y + a) dy \\ &= 0 \quad (f(-y + a) = f(y + a)) \end{aligned}$$

b. For $f(x) = e^{-x}$, $\mu_1 = \mu_2 = 1$, therefore $\alpha_3 = \mu_3$.

$$\begin{aligned} \mu_3 &= \int_0^{\infty} (x - 1)^3 e^{-x} dx = \int_0^{\infty} (x^3 - 3x^2 + 3x - 1) e^{-x} dx \\ &= \Gamma(4) - 3\Gamma(3) + 3\Gamma(2) - \Gamma(1) = 3! - 3 \times 2! + 3 \times 1 - 1 = 3. \end{aligned}$$

c. Each distribution has $\mu_1 = 0$, therefore we must calculate $\mu_2 = EX^2$ and $\mu_4 = EX^4$.

(a)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

Note that it is the density function for standard normal $N(0, 1)$. Thus the *mgf* is given by

$$\begin{aligned} M_x(t) &= e^{ut + \sigma^2 t^2 / 2} = e^{t^2 / 2} \\ \mu_1 &= \frac{d}{dt} M_x(t) |_{t=0} = e^{\frac{t^2}{2}} \cdot t |_{t=0} = 0 \end{aligned}$$

$$\mu_2 = \frac{d^2}{dt^2} M_x(t)|_{t=0} = e^{\frac{t^2}{2}} \cdot (t^2 + 1)|_{t=0} = 1$$

$$\mu_3 = \frac{d^3}{dt^3} M_x(t)|_{t=0} = e^{\frac{t^2}{2}} \cdot (t^3 + 3t)|_{t=0} = 0$$

$$\mu_4 = \frac{d^4}{dt^4} M_x(t)|_{t=0} = e^{\frac{t^2}{2}} (t^4 + 6t^2 + 3)|_{t=0} = 3$$

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{3}{1} = 3$$

(b)

$$f(x) = \frac{1}{2}, \quad -1 < x < 1$$

$$\mu_2 = \int_{-1}^1 x^2 \cdot \frac{1}{2} dx = \frac{x^3}{6} \Big|_{-1}^1 = \frac{1}{3}$$

$$\mu_4 = \int_{-1}^1 x^4 \cdot \frac{1}{2} dx = \frac{x^5}{10} \Big|_{-1}^1 = \frac{1}{5}$$

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{1}{5}}{(\frac{1}{3})^2} = \frac{9}{5}$$

(c)

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

$$\mu_2 = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^0 x^2 e^x dx + \frac{1}{2} \int_0^{\infty} x^2 e^{-x} dx = \int_0^{\infty} x^2 e^{-x} dx = \Gamma(3) = 2$$

$$\mu_4 = \int_{-\infty}^{\infty} x^4 \frac{1}{2} e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^0 x^4 e^x dx + \frac{1}{2} \int_0^{\infty} x^4 e^{-x} dx = \int_0^{\infty} x^4 e^{-x} dx = \Gamma(5) = 24$$

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{24}{4} = 6$$

Thus (c) is most peaked, (a) is next, and (b) is least peaked.

2.32

$$\frac{d}{dt} S(t)|_{t=0} = \frac{d}{dt} \log[M_x(t)]|_{t=0} = \frac{\frac{d}{dt} M_x(t)}{M_x(t)} \Big|_{t=0} = \frac{EX}{1} = EX \quad (\text{since } M_x(0) = Ee^0 = 1)$$

$$\frac{d^2}{dt^2} S(t)|_{t=0} = \frac{d}{dt} \left(\frac{M'_x(t)}{M_x(t)} \right) \Big|_{t=0} = \frac{M_x(t)M''_x(t) - [M'_x(t)]^2}{[M_x(t)]^2} \Big|_{t=0} = \frac{1 \cdot EX^2 - (EX)^2}{1} = \text{Var}X.$$

3.2

Let X = number of defective parts in the sample. Then X is hypergeometric($N = 100, M, K$) where M = number of defectives in the lot and K = sample size.

- a. If there are 6 or more defectives in the lot, then the probability that the lot is accepted ($X = 0$) is at most

$$P(X = 0|N = 100, M = 6, K) = \frac{\binom{6}{0} \binom{94}{K}}{\binom{100}{K}} = \frac{(100 - K) \dots (100 - K - 5)}{100 \dots 95}$$

By trial and error we find $P(X = 0) = 0.10056$ for $K = 31$ and $P(X = 0) = .09182$ for $K = 32$. So the sample size must be at least 32.

- b. Now $P(\text{accept lot}) = P(X = 0 \text{ or } 1)$, and, for 6 or more defectives, the probability is at most

$$P(X = 0 \text{ or } 1|N = 100, M = 6, K) = \frac{\binom{6}{0} \binom{94}{K}}{\binom{100}{K}} + \frac{\binom{6}{1} \binom{94}{K-1}}{\binom{100}{K}}$$

By trial and error we find $P(X = 0 \text{ or } 1) = .10220$ for $K = 50$ and $P(X = 0 \text{ or } 1) = .09331$ for $K = 51$. So the sample size must be at least 51.

3.10

- a. The ways to choose 4 cocaine packets and then 2 nonconcaine packets are $\binom{N}{4} \binom{M}{2}$.

The ways to choose 4 packets and then 2 more parckets are $\binom{N+M}{4} \binom{N+M-4}{2}$.

Hence

$$P(\text{defendant is innocent}) = \frac{\binom{N}{4} \binom{M}{2}}{\binom{N+M}{4} \binom{N+M-4}{2}}$$

- b. Note that $M = 496 - N$, then

$$P_N = \frac{\binom{N}{4} \binom{M}{2}}{\binom{N+M}{4} \binom{N+M-4}{2}} = \frac{\binom{N}{4} \binom{496-N}{2}}{\binom{496}{4} \binom{492}{2}}, \quad N = 4, \dots, 496$$

Since $\frac{P_{N+1}}{P_N} = \frac{\binom{N+1}{4} \binom{496-N-1}{2}}{\binom{N}{4} \binom{496-N}{2}}$, the ratio > 1 for small N_s and the ratio < 1

for large N_s . Compute numerically, we could get $P_{N+1}/P_N = 1 \Leftrightarrow N \approx 330.33$. By trial and error we find

$$P_{330} = \frac{\binom{330}{4} \binom{166}{2}}{\binom{496}{4} \binom{492}{2}} = 0.0220809$$

$$P_{331} = \frac{\binom{331}{4} \binom{165}{2}}{\binom{496}{4} \binom{492}{2}} = 0.0220817$$

Thus when $N = 331$, P_N attains its maximum and $P_N = P_{331} = 0.0221$.

Additional Problems

1

$$\begin{aligned} \text{Var}[3X(X-1) + 5] &= \text{Var}[3X(X-1)] = E[3X(X-1)]^2 - E^2[(3X(X-1))] \\ &= 9(EX^4 - 2EX^3 + EX^2 - E^2X^2 + 2EX^2EX - E^2X) \end{aligned}$$

2

$$E(\phi) = E \frac{(X - EX)}{\sqrt{\text{Var}X}} = \frac{EX - EX}{\sqrt{\text{Var}X}} = 0$$

$$\text{Var}(\phi) = \text{Var} \frac{(X - EX)}{\sqrt{\text{Var}X}} = \frac{\text{Var}X}{\text{Var}X} = 1$$

3

$$M_y(t) = Ee^{ty} = Ee^{(ax+b)t} = Ee^{atx} \cdot e^{bt} = e^{bt} \cdot Ee^{atx} = e^{bt} M_x(at)$$

4

$$M_u(t) = \frac{e^t - 1}{t},$$

$$M_v(t) = \frac{1}{1-t}, \quad t < 1.$$

Let $X \equiv \frac{u-Eu}{\sqrt{\text{Var}u}} = \sqrt{12}u - \sqrt{3}$, $Y \equiv \frac{v-Ev}{\sqrt{\text{Var}v}} = v - 1$. Using the result of equation 3, we have⁵

$$M_X(t) = \frac{e^{\sqrt{3}t} - e^{-\sqrt{3}t}}{2\sqrt{3}t},$$

$$M_Y(t) = \frac{e^{-t}}{1-t}, \quad t < 1.$$

