STAT542 HW3 SOLUTION

Prob 2.2

Apply Thrm 2.1.5

a.

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= f_X(\sqrt{y}) \left| \frac{d}{dy} \sqrt{y} \right| \\
&= 1 \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}, \qquad 0 < y < 1
\end{aligned}
$$

b.

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= f_X(e^{-y}) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= (\frac{n+m+1)!}{n!m!} (e^{-y})^n (1-e^{-y})^m |-e^{-y}|, \qquad 0 < e^{-y} < 1 \\
&= (\frac{n+m+1)!}{n!m!} e^{-y(n+1)} (1-e^{-y})^m, \qquad y > 0
\end{aligned}
$$

c.

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= f_X(\ln y) \left| \frac{d}{dy} \ln y \right| \\
&= \frac{1}{\sigma^2} (\ln y) e^{(-\ln y/\sigma)^2/2} \left| \frac{1}{y} \right|, \qquad \ln y > 0 \\
&= \frac{\ln y}{\sigma^2 y} e^{-\frac{1}{2}\left(\frac{\ln y}{\sigma}\right)^2}, \qquad y > 1
\end{aligned}
$$

Prob 2.3

$$\begin{aligned}
&P(Y=y)\\
&=P(\frac{X}{X+1}=y)\\
&=P(X=\frac{y}{1-y})\\
&=\frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, \qquad y=0,\frac{1}{2},\frac{2}{3},\frac{3}{4},...
\end{aligned}$$

Prob 2.6

Apply Thrm 2.18

a

let $A_0=\{0\}$,

$$A_1=(-\infty,0), \quad g_1(x)=-x^3, \quad g_1^{-1}(y)=-y^{\frac{1}{3}}, \quad y>0$$
$$A_2=(0,\infty), \quad g_2(x)=x^3, \quad g_2^{-1}(y)=y^{\frac{1}{3}}, \quad y>0$$

$$\begin{aligned}
f_Y(y) &= f_X(-y^{\frac{1}{3}})\left|\frac{d}{dy}\left(-y^{\frac{1}{3}}\right)\right|+f_X\left(y^{\frac{1}{3}}\right)\left|\frac{d}{dy}y^{\frac{1}{3}}\right|\\
&= \frac{1}{2}e^{-y^{\frac{1}{3}}}\frac{1}{3}y^{-\frac{2}{3}}+\frac{1}{2}e^{-y^{\frac{1}{3}}}\frac{1}{3}y^{-\frac{2}{3}}\\
&= \frac{1}{3}e^{-y^{\frac{1}{3}}}y^{-\frac{2}{3}}, \qquad y>0
\end{aligned}$$

$\int_{-\infty}^{\infty} f_Y(y)\,dy=\int_0^{\infty}\frac{1}{3}e^{-y^{\frac{1}{3}}}y^{-\frac{2}{3}}\,dy=1$

b

let $A_0=\{0\}$,

$$A_1=(-1,0), g_1(x)=1-x^2, g_1^{-1}(y)=-\sqrt{1-y}, 0<y<1$$
$$A_2=(0,1), g_2(x)=1-x^2, g_2^{-1}(y)=\sqrt{1-y}, 0<y<1$$

$$\begin{aligned}
f_Y(y) &= f_X(-\sqrt{1-y})\left|\frac{d}{dy}(-\sqrt{1-y})\right|+f_X(\sqrt{1-y})\left|\frac{d}{dy}\sqrt{1-y}\right|\\
&= \frac{3}{8}(1-\sqrt{1-y})^2\frac{1}{2\sqrt{1-y}}+\frac{3}{8}(1+\sqrt{1-y})^2\frac{1}{2\sqrt{1-y}}\\
&= \frac{3}{8}\frac{2-y}{\sqrt{1-y}}, \qquad 0<y<1
\end{aligned}$$

$\int_{-\infty}^{\infty} f_Y(y)\,dy=\int_0^1\frac{3}{8}\frac{2-y}{\sqrt{1-y}}\,dy=\frac{3}{8}\int_0^1\sqrt{1-y}\,dy+\frac{3}{8}\int_0^1\frac{1}{\sqrt{1-y}}\,dy=1$

c

Let $A_0 = \{0\}$,

$$A_1 = (-1, 0), g_1(x) = 1 - x^2, g_1^{-1}(y) = -\sqrt{1-y}, 0 < y < 1$$
$$A_2 = (0, 1), g_2(x) = 1 - x, g_2^{-1}(y) = 1 - y, 0 < y < 1$$

$$\begin{aligned} f_Y(y) &= f_X(-\sqrt{1-y}) \left| \frac{d}{dy}(-\sqrt{1-y}) \right| + f_X(1-y) \left| \frac{d}{dy}(1-y) \right| \\ &= \frac{3}{8}(1-\sqrt{1-y})^2 \frac{1}{2\sqrt{1-y}} + \frac{3}{8}(2-y)^2 \\ &= \frac{3(1-\sqrt{1-y})^2}{16\sqrt{1-y}} + \frac{3}{8}(2-y)^2, \qquad 0 < y < 1 \end{aligned}$$

$\int_{-\infty}^{\infty} f_Y(y)\, dy = \int_0^1 \frac{3(1-\sqrt{1-y})^2}{16\sqrt{1-y}}\, dy + \int_0^1 \frac{3}{8}(2-y)^2\, dy = 1$

2.9

From the probability integral transformation, Thrm 2.1.10, we know that if $u(x) = f_X(x)$, then $F_X(x)$ is distributed as $unif(0, 1)$. Therefore, for the given pdf, calculate,

$$u(x) = f_X(x) = \begin{cases} 0, & x < 0 \\ \int_1^x \frac{t-1}{2} dt = \frac{1}{4}(x-1)^2, & 1 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

2.17

a.

$$\begin{aligned} \int_{-\infty}^{m} f(x)dx &= \int_{m}^{\infty} f(x)dx \\ \int_0^m 3x^2 dx &= \int_m^1 3x^2 dx \\ x^3|_0^m &= x^3|_m^1 \\ m^3 &= 1 - m^3 \\ m &= \left(\frac{1}{2}\right)^{\frac{1}{3}} \end{aligned}$$

b.

The function is symmetric about zero, therefore $m = 0$ as long as the integral is finite.

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{1}{\pi} \tan^{-1}(x)|_{-\infty}^{\infty} = \frac{1}{\pi}(\pi/2 + \pi/2) = 1 \quad (1)$$

This is the Cauchy pdf.

2.23

a.

Let $A_0 = \{0\}$,

$$A_1 = (-\infty, 0), g_1(x) = x^2, g_1^{-1}(y) = -\sqrt{y}, 0 < y < 1$$
$$A_2 = (0, \infty), g_2(x) = x^2, g_2^{-1}(y) = \sqrt{y}, 0 < y < 1$$

$$\begin{aligned} f_Y(y) &= f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| + f_X(\sqrt{y}) \left| \frac{d}{dy}\sqrt{y} \right| \\ &= \frac{1}{2}(1-\sqrt{y})\frac{1}{2\sqrt{y}} + \frac{1}{2}(1+\sqrt{y})\frac{1}{2\sqrt{y}} \\ &= \frac{1}{2\sqrt{y}}, \qquad 0 < y < 1 \end{aligned}$$

b.

$$\begin{aligned} EY &= \int_0^1 y f_Y(y)\, dy \\ &= \int_0^1 y \frac{1}{2\sqrt{y}}\, dy \\ &= \frac{1}{3} y^{3/2}|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} EY^2 &= \int_0^1 y^2 f_Y(y)\, dy \\ &= \frac{1}{5} y^{5/2}|_0^1 \\ &= \frac{1}{5} \end{aligned}$$

$$VarY = \frac{1}{5} - (\frac{1}{3})^2 = \frac{4}{45}$$

2.24

a.

$$\begin{aligned} EX &= \int_0^1 x f_X(x)\,dx \\ &= \int_0^1 xa\,x^{a-1}\,dx \\ &= \frac{ax^{a+1}}{a+1}|_0^1 \\ &= \frac{a}{a+1} \end{aligned}$$

$$\begin{aligned} EX^2 &= \int_0^1 x^2 f_X(x)\,dx \\ &= \int_0^1 x^2 a\,x^{a-1}\,dx \\ &= \frac{ax^{a+2}}{a+2}|_0^1 \\ &= \frac{a}{a+2} \end{aligned}$$

$$VarX = \frac{a}{a+2} - (\frac{a}{a+1})^2 = \frac{a}{(a+2)(a+1)^2}$$

b.

$$\begin{aligned} EX &= \sum_1^\infty x\frac{1}{n} \\ &= \frac{1}{n}.\frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

$$\begin{aligned} EX^2 &= \sum_1^\infty x^2\frac{1}{n} \\ &= \frac{1}{n}.\frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6} \end{aligned}$$

$$VarX = \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 = \frac{n^2-1}{12}$$

c.

$$
\begin{aligned}
EX &= \int_0^2 x f_X(x)\,dx \\
&= \int_0^2 x\frac{3}{2}(x-1)^2\,dx \\
&= \frac{3}{2}\int_0^2 (x^3-2x^2+x)\,dx \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
EX^2 &= \int_0^1 x^2 f_X(x)\,dx \\
&= \int_0^2 x^2\frac{3}{2}(x-1)^2\,dx \\
&= \frac{3}{2}\int_0^2 (x^4-2x^3+x^2)\,dx \\
&= \frac{8}{5}
\end{aligned}
$$

$$VarX = \frac{8}{5} - 1 = \frac{3}{5}$$

Additional Problem

i)

$\forall M > 0, \exists$a subset A=$[-\sqrt{M+1}, \sqrt{M+1}] \in R$ such that $\int_{-\infty}^{\infty}|g(x)|dx > \int_{-\sqrt{M+1}}^{\sqrt{M+1}}|g(x)|dx = 2\int_0^{\sqrt{M+1}} xdx = M+1 > M$ Therefore, $\int_{-\infty}^{\infty}|g(x)|dx$ is not absolutely convergent.

ii)

$\forall K \in R$, let $a_n = -\sqrt{n+2|K|-K}$ and $b_n = \sqrt{n+2|K|+K}$ then $\lim_{n\to\infty}\int_{a_n}^{b_n} g(x)dx = \lim_{n\to\infty}(b_n^2 - a_n^2) = \lim_{n\to\infty}\frac{(n+2|K|+K)-(n+2|K|-K)}{2} = K$