STAT542 HW12 SOLUTION

5.23

$$
\begin{aligned}
P(Z > z) &= \sum_{x=1}^{\infty} P(Z > z|x)P(X = x) \\
&= P(U_1 > z, ..., U_x > z|x)P(X = x) \\
&= \sum_{x=1}^{\infty} \prod_{i=1}^{x} P(U_i > z)P(X = x) \quad \text{by independence of the } U_i's \\
&= \sum_{x=1}^{\infty} P(U_i > z)^x P(X = x) \\
&= \sum_{x=1}^{\infty} (1 - z)^x \frac{1}{(e - 1)x!} \\
&= \frac{1}{(e - 1)} \sum_{x=1}^{\infty} \frac{(1 - z)^x}{x!} \\
&= \frac{e^{1-z} - 1}{e - 1}, \quad 0 < z < 1
\end{aligned}
$$

5.24

Use $f_X(x) = 1/\theta, F_X(x) = x/\theta, 0 < x < 1$.
Let $Y = X_{(n)}, Z = X_{(1)}$. Then, from Theorem 5.4.6,

$$
f_{Z,Y}(z, y) = \frac{n!}{0!(n - 2!)0!} \frac{1}{\theta} \frac{1}{\theta} \left(\frac{z}{\theta}\right)^0 \left(\frac{y - z}{\theta}\right)^{n-2} \left(1 - \frac{y}{\theta}\right)^0 = \frac{n(n - 1)}{\theta^n}(y - z)^{n-2}, \quad 0 < z < y < \theta.
$$

Now let $W = Z/Y, Q = Y$. Then $Y = Q, Z = WQ$, and $|J| = q$. Therefore,

$$
f_{W,Q}(w, q) = \frac{n(n - 1)}{\theta^n}(q - qw)^{n-2} q = \frac{n(n - 1)}{\theta^n}(1 - w)^{n-2} q^{n-1}, 0 < w < 1, 0 < q < \theta.
$$

The joint pdf factors into functions of w and q, hence W and Q are independent.

5.27

a. $f_{X(i)|X(j)}(u|v) = \frac{f_{X_{(i)},X_{(j)}}(u,v)}{f_{X_{(j)}}(v)}$. Consider tow cases, depending on which of i or j is greater. Using the formulas from Theorem 5.4.4 and 5.4.6, and after cancellation, we obtain the following.

(i)If $i < j$,

$$\begin{aligned} & f_{X(i)|X(j)}(u|v) \\ = & \frac{(j-1)!}{(i-1)!(j-1-i)!} f_X(u) F_X^{i-1}(u)(F_X(v)-F_X(u))^{j-i-1} F_X^{1-j}(v) \\ = & \frac{(j-1)!}{(i-1)!(j-1-i)!} \frac{f_X(u)}{F_X(v)} \left(\frac{F_X(u)}{F_X(v)}\right)^{i-1} \left(1-\frac{F_X(u)}{F_X(v)}\right)^{j-i-1}, u < v. \end{aligned}$$

Note that this is the pdf of the ith order statistic from a sample of size j-1, from a population with pdf given by the truncated distribution, $f(u) = \frac{f_X(u)}{F_X(v)}, u < v$.

(ii)If $i > j$ and $u > v$,

$$\begin{aligned} & f_{X(i)|X(j)}(u|v) \\ = & \frac{(n-j)!}{(n-1)!(i-1-j)!} f_X(u)(1-F_X(u))^{n-i}(F_X(u)-F_X(v))^{i-1-j}(1-F_X(v))^{j-n} \\ = & \frac{(n-j)!}{(i-j-1)!(n-i)!} \frac{f_X(u)}{1-F_X(v)} \left(\frac{F_X(u)-F_X(v)}{1-F_X(v)}\right)^{i-j-1} (1-\frac{F_X(u)-F_X(v)}{1-F_X(v)})^{n-i}. \end{aligned}$$

This is the pdf of the (i-j)th order statistic from a sample of size n-j, form a population with pdf given by the truncated distribution, $f(u) = \frac{f_X(u)}{1-F_X(v)}, u < v$.

Additional Prob

1

Let $F(X_{(n)}) = U_{(n)} \sim unif(0,1)$, $F(X_{(1)}) = U_{(1)} \sim unif(0,1)$, then the joint pdf of $U_{(n)}$ and $U_{(1)}$ as $f_{(1),(n)}(w,z)$ is follow:

$$\begin{aligned} f_{(1),(n)}(w,z) &= \frac{n!}{(n-2)!} f(w)(F(z)-F(w))^{n-2} f(z) \\ &= n(n-1)(z-w)^{n-2}, \quad 0 < w < z < 1. \\ P((X_{(n)})-(X_{(1)}) \geq p) &= \int_0^{1-p} \int_{w+p}^1 n(n-1)(z-w)^{n-2} dz dw \\ &= \int_0^{1-p} n(n-1) \frac{(z-w)^{n-1}}{n-1} \Big|_{w+p}^1 dw \\ &= \int_0^{1-p} n((1-w)^{n-1} - p^{n-1}) dw \\ &= -n \cdot \frac{(1-w)^n}{n} \Big|_0^{1-p} - np^{n-1}(1-p) \\ &= 1 - p^n - np^{n-1}(1-p) \end{aligned}$$

2

a.

$$\begin{aligned} f(x) &= k\Phi(x)sin^2(x) \\ h(x) &= \Phi(x)sin^2(x) \\ g(x) &= \Phi(x) \\ \frac{h(x)}{g(s)} &= sin^2(x) \leq 1 \Rightarrow M = 1. \end{aligned}$$

Then to generate X from $kh(x)$, conduct the following steps:
1) generate X^* from g.
2) generate $U \sim unif(0,1)$.
3) if $Ug(X^*) < h(X^*)$, then $X = X^*$, else return to 1).

b. To approximate EX^2, we could do the following:
1) use algorithm in a) to generate a large size iid sample $\{X_i\}_{i=1}^n$.
2) compute

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \cdot \frac{h(X_i)}{g(X_i)} = \frac{1}{n}\sum_{i=1}^n X_i^2 sin^2(X_i) \quad (1)$$

$$\frac{1}{n}\sum_{i=1}^n \frac{h(X_i)}{g(X_i)} = \frac{1}{n}\sum_{i=1}^n sin^2(X_i) \quad (2)$$

Then $(1)/(2) \to EX^2 = VarX$ in probability.

c. (First method)
Use empirical cdf to approximate $P(0.3 < X < 1.2)$, that is, $P(0.3 < X < 1.2) = \frac{1}{n}\sum_{i=1}^n I(0.3 < X_i < 1.2)$, where $\{X_i\}_{i=1}^n$ are iid random variables.

(Second method)
Use importance sampling method to do the following:
1) use algorithrm in a) to generate a large size random sample $\{X_i\}_{i=1}^n$.
2) compute

$$\frac{1}{n}\sum_{i=1}^n I(0.3 < X_I < 1.2) \cdot \frac{h(X_i)}{g(X_i)} = \frac{1}{n}\sum_{i=1}^n I(0.3 < X_I < 1.2) sin^2(X_i) \quad (3)$$

$$\frac{1}{n}\sum_{i=1}^n \frac{h(X_i)}{g(X_i)} = \frac{1}{n}\sum_{i=1}^n sin^2(X_i) \quad (4)$$

Then $(3)/(4) \to P(0.3 < X_I < 1.2)$ in probability.

3

$p(\mathbf{x}) \leq 1$, let $h(\mathbf{x}) = I(\mathbf{x} \in [0,1]^5)$ be the uniform density. It could be simulated by drawing 5 iid unif(0,1) coordinates $x_{i.}, 1 \leq i \leq 5$.

Note $1 \cdot h(\mathbf{x}) \geq p(\mathbf{x})$. To get a single realization $\mathbf{X}^*$, do the following:
1) generate $\mathbf{X}^{**} = (X_1^{**}, X_2^{**}, X_3^{**}, X_4^{**}, X_5^{**})$ from h.
2) generate $U \sim unif(0,1)$ which is indepedent from $\mathbf{X}^{**}$.
3) if $Uh(\mathbf{X}^{**}) = U < p(\mathbf{X}^{**})$, set $\mathbf{X}^* = \mathbf{X}^{**}$, otherwise return to 1).