STAT542 HW11 SOLUTION

5.29

Let X_i =weight of ith booklet in package. The $X_i s$ are iid with $EX_i = 1$ and $VarX_i = .05^2$. We want to approximate $P(\sum_{i=1}^{100} X_i > 100.4) = P(\sum_{i=1}^{100} X_i/100 > 1.004) = P(\bar{X} > 1.004)$. By the CLT, $P(\bar{X} > 1.004) \approx P(Z > (1.004-1)/(.05/10)) = P(Z > .8 = .2119)$.

5.36

a.

$$\begin{aligned} EY &= E(E(Y|N)) = E(2N) = 2\theta \\ VarY &= Var(E(Y|N)) + E(Var(Y|N)) = Var(2N) + E(4N) = 8\theta \end{aligned}$$

b.

$$\begin{aligned} M_Y(t) &= Ee^{tY} \\ &= E(E(e^{tY}|N)) \\ &= E\left(\frac{1}{1-2t}\right)^N \qquad M_X(t) = \left(\frac{1}{1-2t}\right)^{p/2} \text{ for } \chi_p^2 \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{1-2t}\right)^n \frac{e^{-\theta}\theta^n}{n!} \\ &= e^{-\theta}\sum_{n=0}^{\infty}\frac{\left(\frac{\theta}{1-2t}\right)^n}{n!} \\ &= e^{-\theta}e^{\frac{\theta}{1-2t}} \\ &= e^{\frac{2t\theta}{1-2t}} \end{aligned}$$

Let $Z = \frac{Y-EY}{\sqrt{Var(Y)}} = \frac{Y-2\theta}{\sqrt{8\theta}}$,

$$\begin{aligned} M_Z(t) &= Ee^{tZ} \\ &= Ee^{t\frac{Y-2\theta}{\sqrt{8\theta}}} \\ &= e^{-\frac{2t\theta}{\sqrt{8\theta}}}E(e^{\frac{tY}{\sqrt{8\theta}}}) \\ &= e^{-\frac{2t\theta}{\sqrt{8\theta}}}M_Y(\frac{t}{\sqrt{8\theta}}) \\ &= e^{-\frac{2t\theta}{\sqrt{8\theta}}}e^{\frac{2t\theta}{\sqrt{8\theta}-2t}} \\ &= e^{\frac{t^2}{2-t\sqrt{2/\theta}}} \\ &\to e^{\frac{t^2}{2}} \qquad \text{as } \theta \to \infty. \end{aligned}$$

Therefore, $\frac{Y-EY}{\sqrt{VarY}} \to N(0,1)$ in distribution as $\theta \to \infty$.

5.42

a.

$$\begin{aligned} P(X_i \le 1-\epsilon) &= \int_0^{1-\epsilon} \beta(1-x)^{\beta-1}dx = 1-\epsilon^\beta \\ P(X_{(n)} \le 1-\epsilon) &= (P(X_i \le 1-\epsilon))^n = (1-\epsilon^\beta)^n \end{aligned}$$

Take $\epsilon = \frac{t}{n^{1/\beta}}$, we have $P(X_{(n)} \le 1 - \frac{t}{n^{1/\beta}}) = (1-(\frac{t}{n^{1/\beta}})^\beta)^n \to 1-exp(-t^\beta)$, which, upon rearranging, yields $P(n^{1/\beta}(1-X_{(n)}) \le t) \to 1-exp(-t^\beta)$in distribution as $n \to \infty$, thus $\nu = 1/\beta$.

b.

$$\begin{aligned} P(X_i \le x) &= 1-e^{-x} \\ P(X_{(n)} \le x) &= (1-e^{-x})^n \end{aligned}$$

Take $a_n = logn$, then

$$\begin{aligned} P(X_{(n)} - a_n \le x) &= P(X_{(n)} \le x + a_n) \\ &= P(X_{(n)} \le x + logn) \\ &= (1-e^{-x-logn})^n \\ &= \left(1 - \frac{e^{-x}}{n}\right)^n \\ &\to exp(-e^{-x}) \quad \text{as } n \to \infty. \end{aligned}$$

5.44

a. $EX_i = p$, $VarX_i = p(1-p)$, $\sqrt{n}(Y_n - p) \to N(0, p(1-p))$ in distribution as $n \to \infty$ by CLT.

b. Let $g(x) = x(1-x)$, $g^{'}(x) = 1-2x$,which exists and is not equal if $x \ne 1/2$. then use Delta method, we have
$\sqrt{n}(g(Y_n) - g(p)) \to N(0, p(1-p)(g^{'}(p))^2)$ in distribution, i.e.,
$\sqrt{n}(Y_n(1-Y_n) - p(1-p)) \to N(0, p(1-p)(1-2p)^2)$ in distribution.

c. Let $g(x) = x(1-x)$, $g^{'}(x) = 1-2x, g^{''}(x) = -2$.
For $x = 1/2, g^{'}(x) = 0$ and $g^{''}(x)$ exists and is not 0.
Then use second-order Delta method, we have
$N(g(Y_n) - g(1/2)) \to -2(1/2)(N(0,1/4))^2$ in distribution, i.e.,
$N(Y_n(1-Y_n) - 1/4) \to -\frac{\chi_1^2}{4}$in distribution.

Additional Prob

1

$$
\begin{aligned}
\sqrt{n}(Y_n - p) &\to N(0, p(1-p)) \text{ in distribution by CLT,} \\
Y_n &\to p \text{ in probability by Weak law of large number,} \\
1/\sqrt{Y_n(1-Y_n)} &\to 1/\sqrt{p(1-p)} \text{ in probability by continuity of the function,} \\
\frac{\sqrt{n}(Y_n - p)}{\sqrt{Y_n(1-Y_n)}} &\to N(0, p(1-p))/\sqrt{p(1-p)} = N(0,1) \text{ in distribution by Slutsky's Theorem.}
\end{aligned}
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2

$$
\begin{aligned}
\text{''} \Rightarrow \text{''} \qquad \forall \epsilon > 0,& \\
P(|X_{ni} - Y_i| \geq \epsilon, \forall i) &\leq P(\|X_n - Y\| \geq \epsilon), \\
&\Rightarrow P(\|X_n - Y\| \geq \epsilon) \to 0, \text{as } X_n \to Y \text{ in probability,} \\
&\Rightarrow P(|X_{ni} - Y_i| \geq \epsilon, \forall i) \to 0, \\
&\Rightarrow X_{ni} \to Y_i, \forall i \text{ in probability.} \\
\text{''} \Leftarrow \text{''} \qquad \forall \epsilon > 0,& \\
P(\|X_n - Y\| \geq \sqrt{k}\epsilon) &\leq P(|X_{ni} - Y_i| \geq \epsilon, \forall i), \\
&\Rightarrow P(|X_{ni} - Y_i| \geq \epsilon, \forall i) \to 0, \text{as } X_{ni} \to Y_i \text{ in probability} \forall i, \\
&\Rightarrow P(\|X_n - Y\| \geq \sqrt{k}\epsilon) \to 0, \\
&\Rightarrow X_n \to Y, \text{ in probability.}
\end{aligned}
$$

3

a.

$$
\begin{aligned}
EX_i &= 3 \\
\bar{X}_n &\to EX_i = 3 \text{ in probability by Weak law of large number.} \\
R_n = \frac{1}{\bar{X}_n} &\to 1/3 \text{ by continuity of the function.}
\end{aligned}
$$

So a=1/3. On the other hand,

$$
\begin{aligned}
E\frac{1}{X_1} &= 1/2 \\
T_n = 1/n \sum_{i=1}^{n} \frac{1}{X_i} &\to E\frac{1}{X_1} = 1/2 \text{ in probability by Weak law of large number.}
\end{aligned}
$$

So b=1/2.

b.

$$\begin{aligned} EX_i &= 3 \\ VarX_i &= 3 \\ \sqrt{n}(\bar{X}_n - 3) &\rightarrow N(0,3) \text{ in distribution by CLT,} \\ \text{Let } g(x) = \frac{1}{x}, & \qquad \text{then } g^{'}(x) = -\frac{1}{x^2}. \\ \text{Use Delta method, } \sqrt{n}(g(\bar{X}_n) - g(3)) &\rightarrow N(0, 3(g^{'}(3))^2) \text{ in distribution ,} \\ i.e., \sqrt{n}(R_n - \frac{1}{3}) &\rightarrow N(0, \frac{1}{27}) \text{ in distribution.} \end{aligned}$$

On the other hand,

$$\begin{aligned} E\frac{1}{X_i} &= \frac{1}{2} \\ Var\frac{1}{X_i} &= \frac{1}{4} \\ \sqrt{n}(T_n - \frac{1}{2}) &\rightarrow N(0, \frac{1}{4}) \text{ in distribution by CLT.} \end{aligned}$$

4

$$\begin{aligned} Y_n \sim Ber(\frac{1}{n}), & \qquad \text{so } P(Y_n = 1) = \frac{1}{n}. \\ &\Rightarrow P(Y_n = 1) \rightarrow 0 \text{ as } n \rightarrow \infty, \\ &\Rightarrow P(Y_n = 0) \rightarrow 1, \\ &\Rightarrow P(nY_n = 0) \rightarrow 1, \\ &\Rightarrow P(X_n = 0) \rightarrow 1, \\ &\Rightarrow \forall \epsilon > 0, P(|X_n| \geq \epsilon) \rightarrow 0, \\ &\Rightarrow X_n \rightarrow 0 \text{ in probability as } n \rightarrow \infty \\ &\Rightarrow X_n \rightarrow 0 \text{ in distribution too as } n \rightarrow \infty. \end{aligned}$$

However, $EX_n = E(nY_n) = n \cdot \frac{1}{n} = 1 \neq 0$.

5

Use Delta method approximation, we have

$$\begin{aligned} EY &= Eg(X) \cong g(EX) = 2, \\ VarY &= Varg(X) \cong Var(X)(g^{'}(EX))^2 = \frac{4}{3}. \end{aligned}$$

Compute EY and VarY directly, we have

$$EY = \int_0^1 \frac{1}{x} dx = lnx|_0^1 = \infty,$$

hence VarY doesn't exist.