STAT542 HW10 SOLUTION

4.63

Since $X = e^Z$ and $g(z) = e^z$ is convex, by Jensens Inequality$EX = Eg(Z) \geq g(EZ) = e^0 = 1$. In fact, there is equality in Jensen's Inequality if and only if there is an interval I wih $P(Z \in I) = 1$ and g(z) is linear on I. But e^z is linear on an interval only if the interval is a singlepoint. So $EX > 1$, unless $P(Z = EZ = 0) = 1$.

5.3

Note that $Y_i \sim$ Bernoulli with $p_i = P(X_i \geq \mu) = 1 - F(\mu)$ for each i. Since the Y_is are iid Bernoulli, $\sum_{i=1}^n Y_i \sim$ binomial$(n, 1 - F(\mu))$.

5.11

Let $g(s) = s^2$. Since $g(.)$ is a convex function, we know from Jensens inequality that$Eg(S) \geq g(ES)$, which implies $\sigma^2 = ES^2 \geq (ES)^2$. Taking square roots, $\sigma \geq ES$. It is clear that the inequality will be strict unless there is an interval I such that g is linear on I and $P(X \in I) = 1$. Since s^2 is linear only on single points,we have $ET^2 > (ET)^2$ for any random variable T, unless $P(T = ET) = 1$.

5.12

$$
\begin{aligned}
E\bar{Y_1} &= E\left|\bar{X}\right| \\
&= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi/n}} e^{-\frac{nx^2}{2}} dx \quad since \left|\bar{X}\right| \sim N(0, 1/n) \\
&= 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi/n}} e^{-\frac{nx^2}{2}} dx \\
&= \sqrt{\frac{2}{n\pi}} \\
E\left|X_i\right| &= \int_{-\infty}^{\infty} |x_i| \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} dx_i \\
&= \frac{1}{\sqrt{2\pi}} \cdot 2\int_0^{\infty} x_i e^{-\frac{x_i^2}{2}} dx_i \\
&= \sqrt{\frac{2}{\pi}} \\
E\bar{Y_2} &= \frac{1}{n}\sum_1^n E\left|X_i\right| = \sqrt{\frac{2}{\pi}}
\end{aligned}
$$

Thus $E\bar{Y_1} < E\bar{Y_2}$.

5.13

$$
\begin{aligned}
E(c\sqrt{S^2}) &= c\sqrt{\frac{\sigma^2}{n-1}} E\left(\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right) \\
&= c\sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \sqrt{q} \frac{1}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} q^{\frac{n-1}{2}-1} e^{-q/2} dq \\
& \qquad \sqrt{S^2(n-1)/\sigma^2} \text{ is the square root of a } \chi^2_{n-1} \text{ random variable} \\
&= c\sqrt{\frac{\sigma^2}{n-1}} \frac{\Gamma(\frac{n}{2})2^{n/2}}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \int_0^\infty \frac{1}{\Gamma(\frac{n}{2})2^{n/2}} q^{\frac{n}{2}-1} e^{-q/2} dq \\
&= c\sqrt{\frac{\sigma^2}{n-1}} \frac{\Gamma(\frac{n}{2})2^{n/2}}{\Gamma(\frac{n-1}{2})2^{(n-1)/2}} \qquad \text{use pdf of } \chi^2_n \\
&= c\sigma\sqrt{\frac{2}{n-1}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \equiv \sigma
\end{aligned}
$$

Thus, $c = \sqrt{\frac{n-1}{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}$.

5.16

a. $\sum_{i=1}^{3} \left(\frac{X_i - i}{i}\right)^2 \sim \chi^2_3$.

b. $\frac{X_1 - 1}{\sqrt{\sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2/2}} \sim t_2$.

c. $\left(\frac{X_1 - 1}{\sqrt{\sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2/2}}\right)^2 \sim F_{1,2}$.

Addtional Prob

$$
\begin{aligned}
M_{X_\alpha}(t) &= \left(\frac{1}{1-t}\right)^\alpha, \\
M_{Z_\alpha}(t) &= Ee^{tZ_\alpha} \\
&= Ee^{t\frac{X_\alpha - EX_\alpha}{\sqrt{Var(X_\alpha)}}} \\
&= Ee^{t\frac{X_\alpha - \alpha}{\sqrt{\alpha}}} \\
&= e^{-t\sqrt{\alpha}} M_{X_\alpha}(t/\sqrt{\alpha}) \\
&= e^{-t\sqrt{\alpha}} \left(\frac{1}{1 - t/\sqrt{\alpha}}\right)^\alpha
\end{aligned}
$$

$$
\begin{aligned}
lnM_{X_\alpha}(t) &= -t\sqrt{\alpha} - \alpha ln \left(1 - t/\sqrt{\alpha}\right) \\
&= -t\sqrt{\alpha} - \alpha \left(-\frac{t}{\sqrt{\alpha}} - \frac{t^2}{2\alpha} + R(\frac{t}{\sqrt{\alpha}}) \right) \\
&= \frac{t^2}{2} + \alpha R(\frac{t}{\sqrt{\alpha}}) \\
&\rightarrow \frac{t^2}{2} \qquad as \quad \alpha \rightarrow \infty.
\end{aligned}
$$