Stat 542 Final Exam

December 16, 2005 **Prof. Vardeman**

1. Suppose that $X_1, X_2, \ldots$ are iid random variables with 4th moment. Use the notation $\mathrm{E}\,X_1 = \mu$, $\mathrm{Var}\,X_1 = \sigma^2$, and $\mathrm{E}\left(X_1 - \mu\right)^4 = \gamma$. Let S_n^2 be the usual sample variance of the first n observations.

a) Argue that $Q_n = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 - \sigma^2\right)$ converges in distribution and identify the limit.

Let $\Delta_n = \sqrt{n}\left(S_n^2 - \sigma^2\right) - Q_n$. It turns out that

$$\Delta_n \xrightarrow{P} 0 \qquad (*)$$

b) As part of showing (*), one needs to argue that

$$\frac{n\sqrt{n}}{n-1}\left(\bar{X}_n - \mu\right)^2 \xrightarrow{P} 0 \qquad (**)$$

Carefully argue (**).

c) Use a) and (*) and argue carefully that $\sqrt{n}\left(S_n^2 - \sigma^2\right)$ converges in distribution and identify the limit.

2. Use a "representational definition" of the chi-squared distributions and find a "large ν" approximation for $F_\nu(x)$, the χ^2_ν cdf, in terms of values of $\Phi(\cdot)$, the standard normal cdf. (The first 4 moments of the standard normal distribution are respectively 0,1,0 and 3.)

3. Suppose that $X_1, X_2,$ and X_3 are iid continuous random variables with marginal pdf $f(x)$ and that $X_{(1)} < X_{(2)} < X_{(3)}$ are the corresponding order statistics.

a) If $f = \phi$, the standard normal pdf, argue that the distribution of the median ($X_{(2)}$) is symmetric about 0.

b) If f is the $\mathrm{U}(0,1)$ pdf, set up completely but do not evaluate a double (iterated) integral giving $P\left[X_{(3)} - X_{(1)} > .5\right]$.

4. A random variable X has pdf of the form

$$f(x) = \begin{cases} C\dfrac{1}{x}\exp(-x) & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

for an unknown $C > 0$. Vardeman would like to evaluate $\mathrm{E}\sin(X)$, but doesn't know how to compute it (or C or the cdf of X) directly. However, he has a supply of iid $\mathrm{U}(0,1)$ random variables available. Completely describe an algorithm he can use to compute the expected value via simulation.

5. Moe, Larry and Curly agree to distribute four \$1 bills between them "at random." Moe suggests selecting a split of the money so that every different breakup of the money is equally likely (so, e.g., $\text{Moe} = \$1, \text{Larry} = \$2, \text{Curly} = \$1$ is as likely as $\text{Moe} = \$0, \text{Larry} = \$0, \text{Curly} = \$4$). Larry insists that instead they (one-bill-at-a-time and independently bill-to-bill) decide who gets each bill with equal probabilities. Curly insists that the two proposed methods are equivalent. Is Curly correct? Carefully argue "yes" or "no."

6. Random variables X and Y are jointly continuous, with (joint) pdf

$$f(x,y) = \begin{cases} 2(y-x) & 0 < x < 1 \text{ and } x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

The region in the (x,y)-plane where the pdf is positive is indicated on the "extra" page of this exam.

a) Find a marginal pdf for Y. Be very careful to say for which y any formula you present holds true.

b) What function of X is the random variable $\mathrm{E}[Y \mid X]$? (Give an explicit formula.)

c) Find a joint pdf for the random variables X and $W = Y - X$.

7. Random variables $W_1, W_2,$ and W_3 are independent, with $W_i \sim \mathrm{N}\left(\mu_i, \sigma^2\right)$. For a constant α , $Y_1 = W_1 + \alpha W_2$, $Y_2 = W_2 + \alpha W_3$, and $Y_3 = W_3 + \alpha W_1$. What is the joint distribution of $\mathbf{Y} = \left(Y_1, Y_2, Y_3\right)'$? Argue carefully that your answer is correct.

Figure for Problem 6