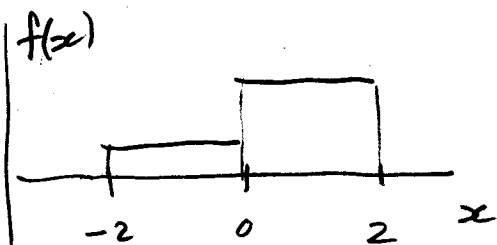


KEY

1. Suppose that  $X$  is a continuous random variable with pdf



$$f_X(x) = \begin{cases} \frac{1}{8} & \text{if } -2 < x < 0 \\ \frac{3}{8} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Evaluate the mean of  $X$ ,  $EX$ .

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f(x) dx = \int_{-2}^0 \frac{x}{8} dx + \int_0^2 \frac{3x}{8} dx \\ &= \left(\frac{1}{8}\right) \frac{x^2}{2} \Big|_{-2}^0 + \left(\frac{3}{8}\right) \frac{x^2}{2} \Big|_0^2 \\ &= \frac{1}{2} \end{aligned}$$

b) Let  $Y = X^2$ . Find a pdf for  $Y$ ,  $f_Y(y)$ .

We must split the set of possible  $x$  into  $(-2, 0)$  and  $(0, 2)$

On  $(-2, 0)$   $g(x) = x^2$  has  $g'(y) = -\sqrt{y}$        $\frac{d}{dy} g^{-1}(y) = -\frac{1}{2\sqrt{y}}$

On  $(0, 2)$   $g(x) = x^2$  has  $g'(y) = \sqrt{y}$        $\frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$

So

$$f_Y(y) = \begin{cases} \frac{1}{8} \left| \frac{-1}{2\sqrt{y}} \right| + \frac{3}{8} \left| \frac{1}{2\sqrt{y}} \right| & 0 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{4\sqrt{y}} & 0 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$W = X \text{ rounded to the nearest integer}$$

( $W$  is a discrete random variable that might be called a "digitized" or "quantized" or "interval censored" version of  $X$ .)

c) Find a probability mass function for  $W$ ,  $f_W(w)$ . (Record possible values and corresponding probabilities in the table below.)

$w$	$f_W(w)$
-2	$\frac{1}{16}$
-1	$\frac{1}{8}$
0	$\frac{1}{4}$
1	$\frac{3}{8}$
2	$\frac{3}{16}$

$$P[W=-2] = P[-2 < X < -1.5] = \frac{1}{16}$$

$$P[W=2] = P[1.5 < X < 2] = \frac{3}{16}$$

$$P[W=-1] = P[-1.5 < X < -.5] = \frac{1}{8}$$

$$P[W=1] = P[.5 < X < 1.5] = \frac{3}{8}$$

$$P[W=0] = P[-.5 < X < .5] = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$$

d) How does  $\text{Var } W$  compare to  $\text{Var } X = \frac{13}{12}$ ? (Evaluate  $\text{Var } W$ .)

$w$	$f_W(w)$	$w f_W(w)$	$w^2 f_W(w)$
-2	$\frac{1}{16}$	$-\frac{2}{16}$	$\frac{4}{16}$
-1	$\frac{2}{16}$	$-\frac{2}{16}$	$\frac{2}{16}$
0	$\frac{4}{16}$	0	0
1	$\frac{6}{16}$	$\frac{6}{16}$	$\frac{6}{16}$
2	$\frac{3}{16}$	$\frac{6}{16}$	$\frac{12}{16}$

$$\begin{aligned} \text{Var } W &= E W^2 - (E W)^2 \\ &= \frac{24}{16} - \left(\frac{1}{2}\right)^2 \\ &= \frac{20}{16} = \frac{5}{4} = \frac{15}{12} \end{aligned}$$

$$\text{Var } W > \text{Var } X$$

$$E W = \frac{1}{2} \quad \frac{24}{16} = E W^2$$

(note that it is not always the case that digitization leaves a mean unchanged)

2. (Moments and MGF's)

a) Argue carefully that there is no distribution for  $X$  such that

$$EX = 0, EX^2 = 1, EX^3 = 0, \text{ and } EX^4 = 0$$

Note that 
$$\begin{aligned} \text{Var } X^2 &= E(X^2)^2 - (EX^2)^2 \\ &= EX^4 - (EX^2)^2 \end{aligned}$$

With the above "moments" we'd have

$$\text{Var } X^2 = 0 - (1)^2 < 0$$

But variances can't be negative, so this would be a contradiction.

b) Argue carefully that the function  $H(t) = 1 + \frac{t^2}{2}$  is not a moment generating function (that there is NO distribution for  $X$  for which  $M_X(t) = H(t)$  for  $t$  in a neighborhood of 0). You may assume the truth of the result in a) whether or not you were able to prove it.

Note that 
$$H'(0) = \left. \frac{2t}{2} \right|_{t=0} = 0$$

$$H''(0) = \left. 1 \right|_{t=0} = 1$$

$$H'''(0) = 0$$

$$H^{(4)}(0) = 0$$

So  $H$  has derivatives at 0 matching the impossible moment sequence in a).  $H$  therefore can not be a mgf.

3. A crop scientist studies the laying of eggs by a particular insect species on the leaves of plants of a particular species. Suppose that in a certain field, a plant leaf is  $k$  times as likely to be unsuitable for egg laying as it is to be suitable for egg laying. Suppose further that the number of eggs laid on a leaf suitable for egg laying has a Poisson( $\lambda$ ) distribution.

a) A certain leaf is has no eggs on it. What is the conditional probability that the leaf is unsuitable for eggs given this fact?

$$P[\text{unsuitable} \mid 0 \text{ eggs}] = \frac{P[0 \text{ eggs} \mid \text{unsuitable}] P[\text{unsuitable}]}{P[0 \text{ eggs} \mid \text{unsuitable}] P[\text{unsuitable}] + P[0 \text{ eggs} \mid \text{suitable}] \cdot P[\text{suitable}]}$$

$$= \frac{1 \cdot \frac{k}{k+1}}{1 \cdot \frac{k}{k+1} + \frac{1}{k+1} e^{-\lambda}} = \frac{k}{k + e^{-\lambda}}$$

b) 5 leaves are each found to have eggs on them. What is a reasonable assessment of the (conditional) probability that at least 4 of the 5 each have more than 1 egg on them?

Each leaf with eggs must be suitable so conditioned on the presence of eggs, for a given leaf

$$P[x \text{ eggs} \mid \text{suitable and } x \geq 1] = \frac{1}{1 - e^{-\lambda}} \left( \frac{e^{-\lambda} \lambda^x}{x!} \right) \quad x=1, 2, \dots$$

So

$$P[\text{more than 1 egg} \mid \text{suitable and at least 1 egg}] = 1 - \frac{1}{1 - e^{-\lambda}} \left( \frac{e^{-\lambda} \lambda}{1!} \right) = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

Then with  $Y =$  number with more than 1 egg, use  $Y \sim \text{Binomial}(5, \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}})$  and have

$$\binom{5}{4} \left( \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}} \right)^4 \left( \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} \right) + \binom{5}{5} \left( \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}} \right)^5$$

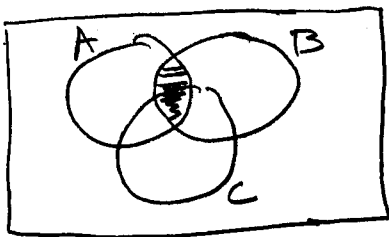
4. A certain security system has passwords that consist of 6 characters. Requirements are that passwords must contain 4 letters (from the 26 letters "a" through "z") with repeats allowed and 2 digits (from the 10 digits "0" through "9"), again with repeats allowed. Passwords are "case-sensitive" so that "a" and "A" are different, and requirements are at least one lower case letter must be employed and at least one capital must be employed. (Of course, order in which the characters appear in the passwords matter.) How many different passwords are there in this system?

$$\# \text{ of passwords} = (\# \text{ with 1 cap letter}) + (\# \text{ with 2 cap letters}) + (\# \text{ with 3 cap letters})$$

Think of choosing in turn 1) positions of the digits, 2) the digits, 3) positions of the caps, 4) the caps and 5) the lower case letters

$$\# \text{ of passwords} = \binom{6}{2} 10^2 \binom{4}{1} 26 \cdot 26^3 + \binom{6}{2} 10^2 \binom{4}{2} 26^2 \cdot 26^2 + \binom{6}{2} 10^2 \binom{4}{3} 26^3 \cdot 26$$

5. Suppose  $P(A) = .2$ ,  $P(B) = .5$ ,  $P(C) = .4$ , and  $P(A^c \cap B \cap C) = .04$ . Suppose further that  $A$  is independent of  $B$  and that  $A \cap B$  is independent of  $C$ . Is  $B$  independent of  $C$ ? Answer yes or no and argue very carefully for the correctness of your answer. (A Venn diagram may help.)



Independence of  $A, B \Rightarrow$   
 $P[A \cap B] = P[\text{shaded area}] = .1 = (.2)(.5)$

Independence of  $A \cap B, C \Rightarrow$   
 $P[(A \cap B) \cap C] = P[\text{diagonal shaded area}] = (.1)(.4) = .04$

$$P[B \cap C] = P[A \cap B \cap C] + P[A^c \cap B \cap C] = .04 + .04 = .08$$

$$P[B] \cdot P[C] = (.5)(.4) = .2 \neq .08$$

No, these 2 events are not independent.