KEY

Stat 542 Exam 2

November 8, 2005 Prof. Vardeman

1. Random variables X and Y are jointly continuous, with (joint) pdf

$$f(x,y)=\begin{cases} y\mathrm{e}^{-y(1+x)} & \text{if } x>0 \text{ and } y>0 \\ 0 & \text{otherwise} \end{cases}$$

8pts a) What are the conditional distributions of $X \mid Y=y$ and of $Y \mid X=x$? (You should NOT have to do any calculus to identify these.)

$X \mid Y=y$: As a function of x, for fixed y, $f(x,y)$ is proportional to e^{-yx} for $x>0$

So given $Y=y$, X is Exponential with mean $\frac{1}{y}$

$Y \mid X=x$: As a function of y, for fixed x, $f(x,y)$ is proportional to $ye^{-(1+x)y}$ for $y>0$

So given $X=x$, Y is $\Gamma\left(2, \frac{1}{1+x}\right)$

6pts b) What is the marginal distribution of Y? For $y>0$

$$f_Y(y)=\int_0^\infty f(x,y)\,dx = ye^{-y}\int_0^\infty e^{-yx}\,dx$$

$$= ye^{-y}\left(\frac{1}{y}\right) = e^{-y}$$

That is, marginally Y is Exponential with mean 1

6pts c) $\mathrm{E}X$ does not exist. Carefully argue this.

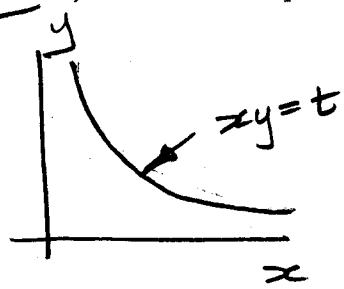
If EX exists, it can be computed using iterated expectations, i.e. if it exists

$$EX = E\,E[X \mid Y] = E\frac{1}{Y} = \int_0^\infty \frac{1}{y}e^{-y}\,dy$$

But $\int_0^\infty \frac{1}{y}e^{-y}\,dy \geq \int_1^\infty \frac{1}{y}e^{-y}\,dy + \int_0^1 \frac{1}{y}e^{-1}\,dy$ and the 2nd of these integrals diverges — So EX does not exist

6pts d) For $t > 0$ completely set up but do not evaluate a double (iterated) integral giving

$$P[XY \le t]$$



$$P[XY \le t] = \int_0^\infty \int_0^{t/y} y e^{-y(1+x)} \, dx \, dy$$

14pts e) Argue carefully that the random variables XY and Y are independent.

Find the joint pdf for $U = XY$ and $V = Y$

$$\underbrace{\begin{aligned} u &= xy \\ v &= y \end{aligned}}_{h} \qquad \underbrace{\begin{aligned} x &= \frac{u}{v} \\ y &= v \end{aligned}}_{h^{-1}}$$

h maps $(0,\infty)^2$ 1-1 onto $(0,\infty)^2$ in a cont$^{\underline{ly}}$ differentiable fashion – $J_{h^{-1}}(u,v) = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{pmatrix} = \frac{1}{v}$

So a joint pdf for (U, V) is

$$f_{U,V}(u,v) = \begin{cases} f_{X,Y}\left(\frac{u}{v}, v\right) \left|\frac{1}{v}\right| & u>0, v>0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} v \exp\left(-v\left(1+\frac{u}{v}\right)\right) \frac{1}{v} & u>0, v>0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \exp(-v-u) & u>0, v>0 \\ 0 & \text{otherwise} \end{cases}$$

U and V are independent Exponential (1) r.v.'s

2. Below is a table specifying a joint pmf for random variables X and Y. Use it on this page.

$y \backslash x$	-2	-1	0	1	2
5	.1				.1
4	.1				.1
3					
2		.1		.1	
1		.1	.1	.1	
0			.1		

6pts a) Are the random variables X and Y independent? Explain.

Clearly not. For example,

$P[X=2]=.2, P[Y=0]=.1$, $P[X=2 \text{ and } Y=0]=0$

but $(.2)(.1) \neq 0$

6pts b) Give a pmf for the random variable $Y - X^2$ in tabular form.

Notice that with probability $(.1)+(.1)+(.1)+(.1)+(.1)=.5$ $Y=X^2$ and that with probability $.5$ $Y=X^2+1$

So the pmf for $W=Y-X^2$ is

w	$f_W(w)$
0	.5
1	.5

8pts c) The random variables $E[Y \mid X]$ and $Var[Y \mid X]$ are functions of the random variable X. Give simple formulas for these random variables in this case.

Note that $E[Y \mid X=x] = \frac{1}{2}x^2 + \frac{1}{2}(x^2+1) = x^2 + \frac{1}{2}$

and that $Var[Y \mid X=x] = \frac{1}{2}\left((x^2+1)-(x^2+\frac{1}{2})\right)^2 + \frac{1}{2}\left(x^2-(x^2+\frac{1}{2})\right)^2 = \frac{1}{4}$

So $E[Y \mid X] = X^2 + \frac{1}{2}$ $\quad Var[Y \mid X] = \frac{1}{4}$

3. Consider six independent mean 0 normal random variables $\tau_1, \tau_2, \varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4. Suppose that the τ's have variance σ_τ^2 and that that ε's have variances σ^2. For a constant μ, define

$$Y_1 = \mu + \tau_1 + \varepsilon_1,\ Y_2 = \mu + \tau_1 + \varepsilon_2,\ Y_3 = \mu + \tau_2 + \varepsilon_3, \text{ and } Y_4 = \mu + \tau_2 + \varepsilon_4$$

(this is a simple case of a so called "random effects model" of applied statistics).

7pts a) What are $\operatorname{Var} Y_1$ and $\operatorname{Cov}(Y_1, Y_2)$?

$$\operatorname{Var} Y_1 = \operatorname{Var}(\mu + \tau_1 + \epsilon_1) = \operatorname{Var}(\tau_1 + \epsilon_1) \overset{\text{independence}}{=} \sigma_\tau^2 + \sigma^2$$

$$\operatorname{Cov}(Y_1, Y_2) = E(Y_1 - \mu)(Y_2 - \mu) = E(\tau_1 + \epsilon_1)(\tau_1 + \epsilon_2) = E\tau_1^2 + \tau_1\epsilon_1 + \tau_1\epsilon_2 \overset{\text{independence and mean } 0}{=} \sigma_\tau^2$$

6pts b) What is the (joint) distribution of $\mathbf{Y}' = (Y_1, Y_2, Y_3, Y_4)$?

The Y_i's are all linear combinations of the same set of independent normal variables (plus μ, of course). There joint dsn is thus MVN_4. Note that (Y_1, Y_2) and (Y_3, Y_4) are independent.

Thus,

$$EY = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \quad \operatorname{Cov}(Y) = \begin{pmatrix} \sigma_\tau^2 + \sigma^2 & \sigma_\tau^2 & 0 & 0 \\ \sigma_\tau^2 & \sigma_\tau^2 + \sigma^2 & 0 & 0 \\ 0 & 0 & \sigma_\tau^2 + \sigma^2 & \sigma_\tau^2 \\ 0 & 0 & \sigma_\tau^2 & \sigma_\tau^2 + \sigma^2 \end{pmatrix}$$

7pts c) Argue carefully that the 3 random variables $Y_1 + Y_2 + Y_3 + Y_4$, $Y_1 - Y_2$, and $Y_3 - Y_4$ are independent.

These 3 are MVN with covariance matrix

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{A} \operatorname{Cov}(Y) \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}}_{A'}$$

Now Then

$$A \operatorname{Cov}(Y) = \begin{pmatrix} 2\sigma_\tau^2 + \sigma^2 & 2\sigma_\tau^2 + \sigma^2 & 2\sigma_\tau^2 + \sigma^2 & 2\sigma_\tau^2 + \sigma^2 \\ \sigma^2 & -\sigma^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & -\sigma^2 \end{pmatrix}$$

So that the covariance matrix is

$$\begin{pmatrix} 8\sigma_\tau^2 + 4\sigma^2 & 0 & 0 \\ 0 & 2\sigma^2 & 0 \\ 0 & 0 & 2\sigma^2 \end{pmatrix}$$

which is clearly <u>diagonal</u>

4. Miscellaneous MGF manipulations.

a) Suppose that $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ has (joint) MGF $M_{\mathbf{X}}(t_1, t_2)$.

4pts i) Let $M_{X_1}(t)$ be the MGF of X_1. It can be written in terms of $M_{\mathbf{X}}$. Do this.

$$\underset{\underset{M_{X_1}(t)}{\parallel}}{E e^{tX_1}} = E e^{tX_1 + 0X_2} = M_X(t, 0)$$

4pts ii) $M_Y(t)$ be the MGF of $Y = X_1 + X_2$. It can be written in terms of $M_{\mathbf{X}}$. Do this.

$$M_Y(t) = E e^{tY} = E e^{t(X_1 + X_2)} = E e^{tX_1 + tX_2}$$
$$= M_X(t, t)$$

b) $M_1(t) = \exp\left(\frac{t^2}{2}\right)$ is the standard normal MGF and $M_2(t) = \frac{\exp(t) - \exp(-t)}{2t}$ is the $U(-1,1)$ MGF.

6pts i) $H(t) = \dfrac{\exp\left(\frac{t^2}{2} + t\right) - \exp\left(\frac{t^2}{2} - t\right)}{2t}$ is a MGF. Carefully argue this.

$H(t) = M_1(t) M_2(t)$ It is the MGF of a sum of two independent r.v.'s The first that is std normal and the second that is $U(-1,1)$

6pts ii) $K(t) = \dfrac{\exp\left(-\frac{t^2}{2} + t\right) - \exp\left(-\frac{t^2}{2} - t\right)}{2t}$ is NOT a MGF. Carefully argue this.

$K(t) = M_2(t)/M_1(t)$ If it were a mgf, we could write $M_1(t) K(t) = M_2(t)$ and have that the dsn of W with mgf K independent of Z std normal would have $W + Z \sim U(-1,1)$ But then $Var(W+Z)$ would have to be $\frac{2^2}{12} = \frac{1}{3}$. while $Var(W+Z) = Var\, W + 1$ This is a contradiction.