Stat 542 Exam 2

November 8, 2005 **Prof. Vardeman**

1. Random variables X and Y are jointly continuous, with (joint) pdf

$$f(x,y) = \begin{cases} y\,e^{-y(1+x)} & \text{if } x>0 \text{ and } y>0 \\ 0 & \text{otherwise} \end{cases}$$

a) What are the conditional distributions of $X \mid Y = y$ and of $Y \mid X = x$? (You should NOT have to do any calculus to identify these.)

$X \mid Y = y$:

$Y \mid X = x$:

b) What is the marginal distribution of Y?

c) $\mathrm{E}\,X$ does not exist. Carefully argue this.

d) For $t > 0$ completely set up but do not evaluate a double (iterated) integral giving

$$P\left[XY \le t\right]$$

e) Argue carefully that the random variables XY and Y are independent.

2. Below is a table specifying a joint pmf for random variables X and Y. Use it on this page.

$y \backslash x$	-2	-1	0	1	2
5	.1				.1
4	.1				.1
3					
2		.1		.1	
1		.1	.1	.1	
0			.1		

a) Are the random variables X and Y independent? Explain.

b) Give a pmf for the random variable $Y - X^2$ in tabular form.

c) The random variables $\mathrm{E}[Y \mid X]$ and $\mathrm{Var}[Y \mid X]$ are functions of the random variable X. Give simple formulas for these random variables in this case.

3. Consider six independent mean 0 normal random variables $\tau_1, \tau_2, \varepsilon_1, \varepsilon_2, \varepsilon_3,$ and ε_4. Suppose that the τ's have variance σ_τ^2 and that that ε's have variances σ^2. For a constant μ, define

$$Y_1 = \mu + \tau_1 + \varepsilon_1,\ Y_2 = \mu + \tau_1 + \varepsilon_2,\ Y_3 = \mu + \tau_2 + \varepsilon_3, \text{ and } Y_4 = \mu + \tau_2 + \varepsilon_4$$

(this is a simple case of a so called "random effects model" of applied statistics).

a) What are $\text{Var}\, Y_1$ and $\text{Cov}(Y_1, Y_2)$?

b) What is the (joint) distribution of $\mathbf{Y}' = (Y_1, Y_2, Y_3, Y_4)$?

c) Argue carefully that the 3 random variables $Y_1 + Y_2 + Y_3 + Y_4$, $Y_1 - Y_2$, and $Y_3 - Y_4$ are independent.

4. Miscellaneous MGF manipulations.

a) Suppose that $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ has (joint) MGF $M_{\mathbf{X}}(t_1, t_2)$.

i) Let $M_{X_1}(t)$ be the MGF of X_1. It can be written in terms of $M_{\mathbf{X}}$. Do this.

ii) $M_Y(t)$ be the MGF of $Y = X_1 + X_2$. It can be written in terms of $M_{\mathbf{X}}$. Do this.

b) $M_1(t) = \exp\left(\frac{t^2}{2}\right)$ is the standard normal MGF and $M_2(t) = \frac{\exp(t) - \exp(-t)}{2t}$ is the $U(-1,1)$ MGF.

i) $H(t) = \frac{\exp\left(\frac{t^2}{2} + t\right) - \exp\left(\frac{t^2}{2} - t\right)}{2t}$ is a MGF. Carefully argue this.

ii) $K(t) = \frac{\exp\left(-\frac{t^2}{2} + t\right) - \exp\left(-\frac{t^2}{2} - t\right)}{2t}$ is NOT a MGF. Carefully argue this.