

## Stat 542 Assignments Fall 2005

### Assignment #1 (Due Friday September 2<sup>nd</sup>)

Required: Problems 1.4, 1.5, 1.13, 1.21, 1.22, 1.33, 1.34, 1.39

Not Required, but Recommended: Problems 1.11, 1.16, 1.20, 1.36, 1.41

### Assignment #2 (Due Friday September 9<sup>th</sup>)

To be turned in: Problems 1.6, 1.7, 1.24, 1.51, 1.53, 1.54, 1.55

Not Required, but Recommended: Problems 1.32, 1.49, 1.52

### Assignment #3 (Due Friday September 16<sup>th</sup>)

To be turned in: Problems 2.2, 2.3, 2.6, 2.9, 2.17, 2.23, 2.24

Also, consider the function  $g(x) = x \forall x \in \mathbb{R}$ . Argue carefully that

i) the integral  $\int_{-\infty}^{\infty} g(x)dx$  is NOT absolutely convergent

ii) for any real constant  $k$ , it is possible by appropriate choice of sequences of constants  $a_n$  and  $b_n$  with  $a_n \rightarrow -\infty$  and  $b_n \rightarrow +\infty$  to make

$$\lim_{n \rightarrow \infty} \int_{a_n}^{b_n} g(x)dx = k$$

(The moral here is that one doesn't want to talk about improper integrals of  $g(x)$ .)

Not Required, but Recommended: Problems 2.12, 2.14, 2.19

### Assignment #4 (Due Friday September 23<sup>rd</sup>)

To be turned in: Problems 2.25, 2.28, 2.32, 3.2, 3.10

Also, do the following:

1) Write an expression for  $\text{Var}(3X(X-1)+5)$  in terms of the moments of  $X$ ,  $E X^r$  for  $r = 1, 2, 3, 4$ .

2) Find the mean and standard deviation (the square root of the variance) of the random variable

$$Y = \frac{X - E X}{\sqrt{\text{Var } X}}$$

3) Suppose that  $X$  has mgf  $M_X(t)$  and define a random variable  $Y = aX + b$  for constants  $a$  and  $b$ . Write an expression for the mgf of  $Y$ ,  $M_Y(t)$ , in terms of  $M_X(t)$ .

4) Suppose that  $U$  has pdf

$$f_U(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

and that  $V$  has pdf

$$f_V(v) = \begin{cases} \exp(-v) & 0 < v \\ 0 & \text{otherwise} \end{cases}$$

Plot on the same set of axes, mgf's for

$$U, V, \frac{U - EU}{\sqrt{\text{Var}U}}, \text{ and } \frac{V - EV}{\sqrt{\text{Var}V}}$$

Not Required, but Recommended: Problems 2.30, 2.38 (note that the answer to part (a) of 2.38 is  $M_X(t) = p^r (1 - (1-p)\exp(t))^{-r}$ )

### **Assignment #5 (Due Friday September 30<sup>th</sup>)**

To be turned in: Problems 3.3, 3.5, 3.7, 3.8a, 3.9, 3.13a

Not Required, but Recommended: Problems 2.4, 2.7a, 3.11a, 3.23, 3.25

### **Assignment #6 (Due Friday October 14<sup>th</sup>)**

To be turned in: Problems 3.17, 3.20, 3.24a,c,e, 3.26, 4.1, 4.5, 4.40, 4.43

Also, derive the standard normal moment generating function and then use your result to verify the form of the (general) normal mgf that Casella and Berger provide on their page 625.

Not Required, but Recommended: Problems 3.16, 3.36, 3.45a, 3.46, 4.4, 4.14, 4.41

### **Assignment #7 (Due Friday October 21<sup>st</sup>)**

To be turned in: Problems 4.7, 4.10, 4.11, 4.22 (use the joint cdf and display 4.1.4), 4.30 (for part b, what is the conditional distribution of the ratio given  $X = x$ , and what then is a joint pdf?), 4.58, 4.59

Also, redo problem 4.43 using the following fact (that is a generalization of the matrix formula for the variance of a linear combination of random variables given in class):

If  $\mathbf{X}$  is  $k \times 1$  with covariance matrix  $\Sigma$ , and  $\mathbf{A}$  is an  $m \times k$  matrix of constants, then the random vector  $\mathbf{Y}$  has covariance matrix  $\mathbf{A}\Sigma\mathbf{A}'$ .

Not Required, but Recommended: Problems 4.13, 4.15, 4.49, 4.51, 4.56

### **Assignment #8 (Due Friday October 28<sup>th</sup>)**

To be turned in: Problems 4.16 (I assume that part c really means pmf, not pdf and I don't follow the note about " $Z = 0$ " or " $X + Y = 0$ "), 4.23, 4.26, 4.28 (skip part c), 4.31, 4.32 (in part b, the distribution is again negative binomial, but not for the same parameters as in a), 4.47, 4.55

Not Required, but Recommended: Problems 4.6, 4.35, 4.36, 4.37

### **Assignment #9 (Due Friday November 4<sup>th</sup>)**

To be turned in: Problems 4.39 (also, in this context, use conditional expected values to evaluate  $E X_1 X_2 X_3$ ), 4.50 and the following:

(Lahiri) For  $(X_1, X_2, X_3)' \sim \text{MVN}_3(\boldsymbol{\mu}, \Sigma)$  with  $\boldsymbol{\mu} = (1, 0, 1)'$  and  $\Sigma = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 10 \end{pmatrix}$ , find

the joint distribution of  $U = X_1 - X_2 + X_3$  and  $V = 3X_1 + X_2 + 1$ .

Let  $X$  and  $Y$  be random variables such that  $X$  is standard normal, the conditional distribution of  $Y$  given  $X = x$  is normal with mean  $x$  variance 2 and  $(X, Y)'$  has a (joint) moment generating function.

Show that  $(X, Y)'$  is bivariate normal with covariance matrix  $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ . (Write the

joint moment generating function as an iterated expectation and evaluate it, finding it to be of precisely the right form.)

What is the distribution of  $\frac{3}{2}X^2 - XY + \frac{1}{2}Y^2$ ?

Is the distribution of  $(X - Y, X + Y)'$  singular, or is it nonsingular? Explain.

Find the mean vector and covariance matrix for a  $k = 4$  Dirichlet random vector, where the parameter vector  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  is  $(1, 4, 2, 3)$ . Is this covariance matrix

nonsingular? (Why is this no surprise to you?) Evaluate  $P[X_1 + X_3 > .5]$ .

Find the mean vector and covariance matrix for a multiple hypergeometric random vector with  $N_1 = 10, N_2 = 40, N_3 = 20$  and  $N_4 = 30$  when  $n = 10$ . How do these compare to the

mean and covariance matrix for a multinomial random vector with  $n = 10$  and  $\mathbf{p} = (.1, .4, .2, .3)$ ? Suppose I desire to find a continuous distribution with the same mean and covariance matrix at least somewhat comparable to that of the multiple hypergeometric. Identify both a multivariate normal distribution and something built from a Dirichlet distribution that I might use. (In the second case, you may simply say you'll use the distribution of some particular multiple of a variable with some particular Dirichlet distribution.)

### Assignment #10 (Due Friday November 18<sup>th</sup>)

To be turned in: Problems 4.63, 5.3, 5.11, 5.12, 5.13, 5.16

Also, suppose that  $X_\alpha \sim \Gamma(\alpha, 1)$  and that  $Z_\alpha = \frac{X_\alpha - E X_\alpha}{\sqrt{\text{Var } X_\alpha}}$ . Let  $M_{Z_\alpha}(t)$  be the moment

generating function for  $Z_\alpha$  (you may take the form of the mgf for  $X_\alpha$  as given).

Compute (for  $t < 1$ )

$$\lim_{\alpha \rightarrow \infty} \ln M_{Z_\alpha}(t)$$

Not Required, but Recommended: Problems 5.2, 5.8, 5.10, 5.15, 5.19, 5.20

### Assignment #11 (Due Friday December 2<sup>nd</sup>)

To be turned in: Problems 5.29, 5.36, 5.42, 5.44 and the following exercises.

- In the context of Problem 5.44, argue carefully that  $\sqrt{n}(Y_n - p)/\sqrt{Y_n(1 - Y_n)}$  converges in distribution to standard normal.
- Prove that if  $X_1, X_2, X_3, \dots$  are random  $k$ -vectors, then  $X_n \xrightarrow{P} Y$  iff  $X_{ni} \xrightarrow{P} Y_i$  for all  $i = 1, 2, \dots, k$ . (You'll probably want to use the key points I laid out in class.)
- (Lahiri) Let  $X_1, X_2, X_3, \dots$  be a sequence of iid Gamma  $(3, 1)$  random variables.
  - Let  $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ ,  $R_n = 1/\bar{X}_n$ , and  $T_n = n^{-1} \sum_{i=1}^n X_i^{-1}$ . Find constants  $a$  and  $b$  so that  $R_n \xrightarrow{P} a$  and  $T_n \xrightarrow{P} b$ . (Argue carefully that your answers are correct.)
  - Find limiting distributions for  $\sqrt{n}(R_n - a)$  and  $\sqrt{n}(T_n - b)$ . (Argue carefully that your answers are correct.)
- (Convergence in probability or in distribution says nothing about moments) Suppose that  $Y_n \sim \text{Bernoulli}\left(\frac{1}{n}\right)$  and take  $X_n = nY_n$ . Show that  $X_n \xrightarrow{P} 0$  (and therefore that  $X_n \xrightarrow{L} 0$ ). Does  $EX_n \rightarrow 0$ ?

5. (The "delta method" can be spectacularly wrong when used as a way of approximating moments) Suppose that  $X \sim U(0,1)$ . Consider the function

$$g(x) = \frac{1}{x} \text{ and the random variable } Y = g(X). \text{ Find the "delta method"}$$

approximation to the mean and variance of  $Y$ . How do they compare to the real mean and variance of  $Y$ ?

Not Required, but Recommended: Problems 5.18, 5.30, 5.33, 5.34

### **Assignment #12 (Not to be Turned in)**

Problems 5.23, 5.24, 5.27a and the following exercises:

1. For  $X_1, X_2, \dots, X_n$  iid with marginal pdf  $f$  and marginal cdf  $F$ , show that

$$P[F(X_{(n)}) - F(X_{(1)}) \geq p] = 1 - p^n - n(1-p)p^{n-1}.$$

2. Suppose for sake of example that I am interested in a continuous distribution with pdf  $f(x) = K\phi(x)\sin^2(x)$  for  $\phi(x)$  the standard normal pdf and  $K$  some appropriate normalizing constant (that to my knowledge can not be evaluated "in closed form" but that MathCad indicates is no more than 2.32).

- Carefully and completely describe a rejection sampling algorithm for generating  $X$  from this distribution.
- Then consider the matter of finding the variance for this distribution. One possibility is simply to generate a large number of iid observations (via the rejection sampling algorithm) and use the sample variance as an approximation for this. Another is to use importance sampling to approximate  $E X^2$  (since by symmetry  $E X = 0$ . Carefully and completely describe how to do this.
- Carefully describe two ways of approximating  $P[.3 < X < 1.2]$  via simulation. (For the first, suppose that  $X_1, X_2, \dots, X_n$  iid  $f$  are available via the rejection algorithm. For the second, consider importance sampling using iid standard normal variables. In the second case, you may want to consider the function  $q(x) = 1$  if  $x \in (.3, 1.2)$  and  $q(x) = 0$  otherwise.)

3. A curious statistician is interested in the properties of a 5-dimensional distribution on  $[0,1]^5$  that has pdf proportional to

$$p(\mathbf{x}) = \exp\left[-\left((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 + (x_5 - x_1)^2\right)\right]$$

This person decides to simulate from this distribution. Carefully describe a rejection algorithm that could be used to produce realizations  $\mathbf{x}$  from this distribution.