Suppose that $X \sim$ Bernoulli $\left(\frac{1}{2}\right)$, $U \sim$ Uniform $(0,1)$ and $Z \sim$ Normal $(0,1)$ are independent random variables. Define

$$W = XU + (1-X)Z$$

(so that W is either U or Z, depending upon the value of X). In your answers to what follows, use Φ for the standard normal cdf and ϕ for the standard normal pdf.

a) Find expressions for

$$P[X = 0 \text{ and } W \leq w]$$

and

$$P[X = 1 \text{ and } W \leq w]$$

(be sure to cover all cases $w \leq 0,\ 0 < w \leq 1$ and $w > 1$).

b) Find expressions for the cdf of W

$$F_W(w) = P[W \leq w]$$

and a pdf for W

$$f_W(w)$$

(again be sure to cover all cases for w). (Hint: Part **a)** is relevant here.)

c) Use the notion of conditioning to find numerical values for the mean and variance of W, $\mathrm{E}W$ and $\mathrm{Var}W$.

d) Evaluate the correlation between W and Z.

e) Consider the function of W

$$h(W) = \begin{cases} 0 & \text{if } W < 0 \text{ or if } W > 1 \\ \frac{1}{1+\phi(W)} & \text{if } 0 < W < 1 \end{cases} .$$

The random variable

$$Xh(W)$$

can be written as a function of X and U. Do this. Then use your expression to write an integral that gives

$$\mathrm{E}Xh(W)$$

(the numerical value of this is .3733 but don't try to evaluate it here).

f) If only W is observable, $h(W)$ is a sensible "predictor" of X. For one thing, it has the same mean as X, namely $\frac{1}{2}$ (you may assume this without proof). Compare the predictors of X,

$$\widehat{X}_1 = h(W)$$

and

$$\widehat{X}_2 = \frac{1}{2}$$

on the basis of their mean squared differences from X (the values $\mathrm{E}\left(X - \widehat{X}_1\right)^2$ and $\mathrm{E}\left(X - \widehat{X}_2\right)^2$).