Throughout this question $f(x) > 0$ will stand for a probability density function (pdf) on $\Re^1$ and $F(x)$ will stand for the corresponding (strictly increasing) cumulative distribution function (cdf).

First, suppose that $X \sim F$ and for $t \in \Re^1$, define

$$Y(t) = \begin{cases} 1 & \text{if } X \leq t \\ 0 & \text{if } X > t \end{cases}$$

a) What is the distribution of $Y(t)$? What are $\mathrm{E}Y(t)$ and $\mathrm{Var}Y(t)$?

b) For $s < u$, evaluate $\mathrm{Cov}(Y(s), Y(u))$. (It may be useful to note that $Y(s) \cdot Y(u) = Y(s)$.)

c) For $s < u$, evaluate $\mathrm{E}(Y(u) - Y(s))$ and $\mathrm{Var}(Y(u) - Y(s))$.

Now suppose that $X_1, X_2, \ldots, X_n$ are iid F and for $t \in \Re^1$, define

$$Y_i(t) = \begin{cases} 1 & \text{if } X_i \leq t \\ 0 & \text{if } X_i > t \end{cases}$$

and $\overline{Y_n}(t) = \frac{1}{n}\sum_{i=1}^{n} Y_i(t)$.

d) For $s < u$, $\overline{Y_n}(u) - \overline{Y_n}(s)$ converges in probability. Identify the limit and argue carefully that this variable converges to that value.

e) Suppose that $s < u$ are such that $F(u) - F(s) = .5$. For $n = 400$ approximate $P[\,|\overline{Y_n}(u) - \overline{Y_n}(s) - .5| < .05]$ and justify your approximation.

f) Evaluate $P[\min X_i < F^{-1}(.5) < \max X_i]$. (This is the probability that the interval $(\min X_i, \max X_i)$ covers the distribution median.)

g) Suppose that $W \sim F$ independent of $X_1, X_2, \ldots, X_n$. Evaluate

$$P[\min X_i < W < \max X_i] \quad .$$

(You may use a simple symmetry argument.)