KEY

Stat 542 Final Exam

December 18, 2001 **Professor Vardeman**

8pts 1. If $X \sim \mathrm{N}(\mu, \sigma^2)$ it is common to say that $\exp(X)$ has the "lognormal" distribution with parameters μ and σ^2. Suppose that $W_1, W_2, \ldots$ are iid lognormal (μ, σ^2).

a) **Argue** very carefully that $V_2 = W_1 \cdot W_2$ has a lognormal distribution and **identify** the parameters for that distribution. If X_1, X_2 are independent $N(\mu,\sigma^2)$ r.v.'s V_2 has the same dsn as $e^{X_1} e^{X_2} = e^{X_1+X_2}$. But $X_1+X_2 \sim N(2\mu, 2\sigma^2)$ and so V_2 is lognormal $(2\mu, 2\sigma^2)$.

9pts b) Let $V_n = W_1 \cdot W_2 \cdot \cdots \cdot W_n$. Both $(V_n)^{\frac{1}{n}}$ and $(V_n)^{\frac{1}{n^2}}$ converge in probability to constants. **Identify** those constants. If $X_1, X_2, \ldots, X_n$ are iid $N(\mu,\sigma^2)$, $\overline{X}_n \xrightarrow{P} \mu$ and so $\frac{1}{n}\overline{X}_n \xrightarrow{P} 0$. V_n has the same dsn as $e^{n\overline{X}_n}$ so that $V_n^{\frac{1}{n}}$ has the dsn of $e^{\overline{X}_n}$ and $\therefore$ $V_n^{\frac{1}{n}} \xrightarrow{d} e^{\mu}$ and $\therefore$ $V_n^{\frac{1}{n}} \xrightarrow{P} e^{\mu}$. $V_n^{\frac{1}{n^2}}$ has the dsn of $e^{\frac{1}{n}\overline{X}_n}$ and $\therefore$ $V_n^{\frac{1}{n^2}} \xrightarrow{P} e^0 = 1$

$(V_n)^{\frac{1}{n}} \xrightarrow{P}$ e^{μ} $(V_n)^{\frac{1}{n^2}} \xrightarrow{P}$ $e^0 = 1$

10pts 2. Suppose that $X \sim t_\nu$. **What** is the distribution of $1/X^2$? (Name a standard distribution including the values of any parameters and **argue carefully** that your answer is correct.)

For Z and W independent, Z standard normal and $W \sim \chi^2_\nu$, X has the dsn of $Z/\sqrt{W/\nu}$. So $1/X^2$ has the dsn of $(W/\nu)/Z^2$. Since the square of a standard normal is χ^2_1, this variable is by definition $F_{\nu,1}$ distributed.

3. Suppose that $X_1, X_2, \ldots, X_n$ are iid $\mathrm{Exp}(\beta)$.

10pts a) Temporarily suppose that $\beta = 1$, and write out completely (but don't try to evaluate) a double integral giving the expected range of these n variables, $\mathrm{E}(X_{(n)} - X_{(1)})$. For $u = x_{(1)}$ and $v = x_{(n)}$ the joint pdf of the min and max is

$$f_{1,n}(u,v) = \begin{cases} \dfrac{n!}{1!(n-2)!1!} e^{-u}\left(e^{-u} - e^{-v}\right)^{n-2} e^{-v} & \text{for } 0 < u < v \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_{(n)} - X_{(1)}) = \int_0^\infty \int_0^v (v-u)\, n(n-1)\, e^{-u-v}\left(e^{-v} - e^{-u}\right) du\, dv$$

An acceptable alternative is to use the fact that $E(X_{(n)} - X_{(1)}) = EX_{(n)} - EX_{(1)}$ and employ the marginals of $X_{(n)}$ and $X_{(1)}$

10pts b) For $n = 100$, approximate $P\left[85\beta < \sum_{i=1}^{100} X_i < 115\beta\right]$ (and carefully justify your answer).

Desired probability $= P[.85\beta < \bar{X}_{100} < 1.15\beta] = P[-.15\beta < \bar{X}_{100} - \beta < .15\beta]$

$= P[-1.5\beta < \sqrt{100}(\bar{X}_{100} - \beta) < 1.5\beta]$

CLT says $\sqrt{n}(\bar{X}_n - \beta) \xrightarrow{\mathcal{L}} N(0, \beta^2)$, so this is approximately

$P[-1.5 < Z < 1.5] = \Phi(1.5) - \Phi(-1.5)$

9pts c) Suppose that $g: (0, \infty) \longrightarrow \Re^1$ is differentiable. What must be true about g in order that $\sqrt{n}\left(g(\bar{X}_n) - g(\beta)\right)$ have the same approximate (large n) distribution for all β? What is a function that satisfies your condition?

The delta method says that the limiting dsn is $g'(\beta) \cdot N(0, \beta^2)$. So we want $(g'(\beta))^2 \beta^2 = \text{constant}$

i.e. we want $(g'(\beta))^2 = \dfrac{\text{constant}}{\beta^2}$. It would suffice to have $g'(\beta) = \dfrac{1}{\beta}$, e.g. $g(\beta) = \ln(\beta)$ works.

Assume $\{x \mid f(x)>0\} = \{x \mid g(x)>0\}$

8pts 4. One measure of the "information" in a distribution with pdf f about a distribution with pdf g is $I = \int -\ln\left[g(x)/f(x)\right] f(x)dx$. Argue carefully that $I \geq 0$. (Hint: For $X \sim f$, and $Y = g(X)/f(X)$ what is EY?)

$$EY = \int \frac{g(x)}{f(x)} f(x)\,dx = \int g(x)\,dx = 1$$

the function $h(t) = -\ln t$ has $h'(t) = -\frac{1}{t}$ and $h''(t) = \frac{1}{t^2} > 0$ and thus is (strictly) convex on $(0,\infty)$ – So Jensen says

$$I = E - \ln\left(\frac{g(X)}{f(X)}\right) \geq -\ln\left(E\frac{g(X)}{f(X)}\right) = -\ln(1) = 0$$

5. A 10-dimensional random vector $\underset{\sim}{X}$ has a complicated (joint) pdf that is proportional to the function

$$h(\underset{\sim}{x}) = \begin{cases} \exp\left(\prod_{i=1}^{10} \sin^2(x_i)\right) & \text{if each } x_i \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

Notice that $h(x) \leq e^1$.

9pts a) Carefully describe an algorithm (using only iid Uniform $(0,1)$ random variables) for generating a (10-dimensional) realization of $\underset{\sim}{X}$. We can use a rejection algorithm with g the uniform density on $(0,1)^{10}$ and $M = e^1$ – That is

1) Generate $U_1, U_2, \ldots, U_{10}$ iid $U(0,1)$ and let $\underset{\sim}{X}^* = (U_1, U_2, \ldots, U_{10})$
2) Generate $U_{11} \sim U(0,1)$
3) If $e \cdot U_{11} < h(\underset{\sim}{X}^*)$ set $\underset{\sim}{X} = \underset{\sim}{X}^*$ otherwise return to 1)

7pts b) Describe any simulation-based method of approximating $\mathrm{E}\,X_1X_2$. (Whether or not you could do part a), you may if you wish presume that you have such an algorithm at your disposal.)

Generate some large number of realizations $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$

Compute $\frac{1}{n}\sum X_{i1}X_{i2}$. WLLN guarantees that this converges in probability to EX_1X_2.

6. Eight perfectly matched teams enter a single-elimination tournament where the teams are placed into the brackets at random. (In any game played, both teams have the same chance of wining. Only teams that win continue to play games. Purely for the sake of absolute clarity, example tournament brackets are attached as a 7th page on this exam.) Name the teams #1 through #8.

7pts a) What is the probability that teams #1 and #2 meet in a first round game?

By symmetry $P[\text{team \#1's opponent is team \#2}] = \frac{1}{7}$

10pts b) What is the probability that teams #1 and #2 meet in either a first round game or in a second round game?

This is $P[\text{team \#1 and team \#2 meet in 1st round}]$
$+ P[\text{team \#1 and team \#2 meet in 2nd round}]$

$P[\text{team \#1 and team \#2 meet in 2nd round}]$
$= P[\text{team \#2 is placed in one of the 2 slots that could meet team \#1 in 2nd round}] \times P[\text{both \#1 and \#2 win 1st round games} \mid \text{placement so 2nd round game is possible}]$
$= \frac{2}{7} \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \frac{1}{14}$

So desired probability $= \frac{1}{7} + \frac{1}{14} = \frac{3}{14}$

9pts 7. Suppose that X_1, X_2, Y_1, Y_2, Y_3 are independent normal random variables with $\mathrm{E}X_i = \mu$ and $\mathrm{Var}\,X_i = \sigma^2$ and $\mathrm{E}Y_j = 0$ and $\mathrm{Var}\,Y_j = \eta^2$. What is the joint distribution of $W_1 = X_1 + Y_1, W_2 = X_1 + Y_2$ and $W_3 = X_2 + Y_3$? (Describe it completely, but you don't need to multiply matrices out.)

$$\begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim MVN_5\left(\begin{pmatrix} \mu \\ \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma\right) \text{ for } \Sigma = \mathrm{diag}(\sigma^2, \sigma^2, \eta^2, \eta^2, \eta^2)$$

Then $W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$ is $A \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$ for $\underset{3\times 5}{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$. So

$W \sim MVN_3\left(A \begin{pmatrix} \mu \\ \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}, A\Sigma A'\right)$ i.e. with mean vector $\begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix}$

and $\mathrm{Cov}\,W = \begin{pmatrix} \sigma^2+\eta^2 & \sigma^2 & 0 \\ \sigma^2 & \sigma^2+\eta^2 & 0 \\ 0 & 0 & \sigma^2+\eta^2 \end{pmatrix}$

8. Suppose that $P[\lambda=1]=.3$ and $P[\lambda=2]=.7$ and that conditional on λ, $X\sim\text{Poisson}(\lambda)$.

9pts a) Evaluate $P[\lambda=1\,|\,X=0]$.

$$P[\lambda=1 \text{ and } X=0]=P[X=0\,|\,\lambda=1]P[\lambda=1]=\frac{e^{-1}(1)^0}{0!}(.3)=.3e^{-1}$$

$$P[X=0]=P[\lambda=1 \text{ and } X=0]+P[\lambda=2 \text{ and } X=0]=.3e^{-1}+.7e^{-2}$$

$$P[\lambda=1\,|\,X=0]=\frac{.3e^{-1}}{.3e^{-1}+.7e^{-2}}=\frac{.3}{.3+.7e^{-1}}=.5381$$

9pts b) Find both $\operatorname{Var}X$ and $\operatorname{Cov}(\lambda,X)$.

$$\operatorname{Var}X=\operatorname{Var}E[X|\lambda]+E\operatorname{Var}[X|\lambda]=\operatorname{Var}\lambda+E\lambda$$
$$=E\lambda^2-(E\lambda)^2+E\lambda$$
$$=4(.7)+(1)(.3)-(1.7)^2+1.7=1.91$$

$$\operatorname{Cov}(\lambda,X)=E\lambda X-E\lambda EX=E\lambda E[X|\lambda]-E\lambda\, EE[X|\lambda]$$
$$=E\lambda^2-(E\lambda)^2$$
$$=4(.7)+1(.3)-(1.7)^2=.21$$

6pts c) Find the moment generating function of X, $M_X(t)$.

$$Ee^{tX}=EE[e^{tX}|\lambda]=Ee^{\lambda(e^t-1)}$$
$$=.3e^{(e^t-1)}+.7e^{2(e^t-1)}$$
$$=e^{(e^t-1)}\left(.3+.7e^{(e^t-1)}\right)$$

9. Random variables X and Y are jointly continuous with pdf

$$f(x,y)=\begin{cases}\frac{3}{2}\left(x^2+y^2\right) & \text{if } 0<x<1 \text{ and } 0<y<1\\ 0 & \text{otherwise}\end{cases}$$

8pts a) For $0<x<1$ and $0<y<1$ find the value of the (joint) cdf $F(x,y)$. For such (x,y)

$$F(x,y) = P[X\le x \text{ and } Y\le y]$$

$$= \int_0^y \int_0^x \tfrac{3}{2}\left(t^2+s^2\right) ds\, dt$$

$$= \tfrac{3}{2}\int_0^y xt^2 + \tfrac{x^3}{3}\, dt = \tfrac{3}{2}\left(\tfrac{xy^3}{3} + \tfrac{yx^3}{3}\right) = \tfrac{xy}{2}\left(y^2+x^2\right)$$

8pts b) Find a marginal pdf for X. (Be careful to specify it completely.)

The cdf of X is $F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ F(x,1) & \text{if } x\in(0,1) \\ 1 & \text{if } x>1\end{cases}$

So $\frac{d}{dx}F_X(x) = \begin{cases}\frac{d}{dx}\left(\frac{x}{2}(1+x^2)\right) & \text{if } x\in(0,1) \\ 0 & \text{otherwise}\end{cases} = \begin{cases}\frac{1}{2}+\frac{3}{2}x^2 & x\in(0,1)\\ 0 & \text{otherwise}\end{cases}$

6pts c) Are X and Y independent? Explain.

no, by symmetry $f_Y(y) = \begin{cases}\frac{1}{2}+\frac{3}{2}y^2 & y\in(0,1)\\ 0 & \text{otherwise}\end{cases}$

and $f(x,y) \neq f_X(x)f_Y(y)$

8pts d) Find a pdf for $W = X^2$. for $w\in(0,1)$

$$P[X^2\le w] = F_X(\sqrt{w}) = \frac{\sqrt{w}}{2}(1+w)$$

so $f_W(w) = \begin{cases}\frac{1}{4\sqrt{w}} + \frac{3}{4}\sqrt{w} & \text{for } w\in(0,1)\\ 0 & \text{otherwise}\end{cases}$

Round

1st 2nd 3rd