Stat 542 Final Exam

December 18, 2001 **Professor Vardeman**

1. If $X \sim \mathrm{N}\left(m, s^2\right)$ it is common to say that $\exp(X)$ has the "lognormal" distribution with parameters m and s^2. Suppose that $W_1, W_2, \ldots$ are iid lognormal $\left(m, s^2\right)$.

a) **Argue** very carefully that $V_2 = W_1 \cdot W_2$ has a lognormal distribution and **identify** the parameters for that distribution.

b) Let $V_n = W_1 \cdot W_2 \cdot \cdots \cdot W_n$. Both $\left(V_n\right)^{\frac{1}{n}}$ and $\left(V_n\right)^{\frac{1}{n^2}}$ converge in probability to constants. **Identify** those constants.

$\left(V_n\right)^{\frac{1}{n}} \xrightarrow{P}$ _______________ $\left(V_n\right)^{\frac{1}{n^2}} \xrightarrow{P}$ _______________

2. Suppose that $X \sim t_n$. **What** is the distribution of $1/X^2$? (**Name** a standard distribution including the values of any parameters and **argue carefully** that your answer is correct.)

3. Suppose that $X_1, X_2, \ldots, X_n$ are iid $\mathrm{Exp}(b)$.

a) Temporarily suppose that $b = 1$, and **write out** completely (but don't try to evaluate) a double integral giving the expected range of these *n* variables, $\mathrm{E}\left(X_{(n)} - X_{(1)}\right)$. (Alternatively, you may write out an appropriate difference in 1-dimensional integrals.)

b) For $n = 100$, **approximate** $P\left[85b < \sum_{i=1}^{100} X_i < 115b\right]$ (and carefully justify your answer).

c) Suppose that $g : (0, \infty) \longrightarrow \Re^1$ is differentiable. **What** must be true about g in order that $\sqrt{n}\left(g\left(\bar{X}_n\right) - g(b)\right)$ have the same approximate (large *n*) distribution for all b? **What** is a function that satisfies your condition?

4. One measure of the "information" in a distribution with pdf f about a distribution with pdf g is $I = \int -\ln\left[g(x)/f(x)\right] f(x)dx$. Assume that $\{x \mid f(x) > 0\} = \{x \mid g(x) > 0\}$ and **argue** carefully that $I \geq 0$. (Hint: For $X \sim f$, and $Y = g(X)/f(X)$ what is $\mathrm{E}Y$?)

5. A 10-dimensional random vector $\underset{\sim}{X}$ has a complicated (joint) pdf that is proportional to the function

$$h(\underset{\sim}{x}) = \begin{cases} \exp\left(\prod_{i=1}^{10} \sin^2(x_i)\right) & \text{if each } x_i \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

Notice that $h(\underset{\sim}{x}) \leq e^1$.

a) Carefully **describe** an algorithm (using only iid Uniform $(0,1)$ random variables) for generating a (10-dimensional) realization of $\underset{\sim}{X}$.

b) **Describe** any simulation-based method of approximating $\mathrm{E}\, X_1 X_2$. (Whether or not you could do part a), you may if you wish presume that you have such an algorithm at your disposal.)

6. Eight perfectly matched teams enter a single-elimination tournament where the teams are placed into the brackets at random. (In any game played, both teams have the same chance of wining. Only teams that win continue to play games. Purely for the sake of absolute clarity, example tournament brackets are attached as a 7th page on this exam.) Name the teams #1 through #8.

a) **What** is the probability that teams #1 and #2 meet in a first round game?

b) **Wha**t is the probability that teams #1 and #2 meet in either a first round game or in a second round game?

7. Suppose that X_1, X_2, Y_1, Y_2, Y_3 are independent normal random variables with $\mathrm{E}X_i = m$ and $\mathrm{Var}\,X_i = s^2$ and $\mathrm{E}Y_j = 0$ and $\mathrm{Var}\,Y_j = h^2$. **What** is the joint distribution of $W_1 = X_1 + Y_1, W_2 = X_1 + Y_2$ and $W_3 = X_2 + Y_3$? (Describe it completely, but you don't need to multiply matrices out.)

8. Suppose that $P[l = 1] = .3$ and $P[l = 2] = .7$ and that conditional on l, $X \sim \text{Poisson}(l)$.

a) **Evaluate** $P[l = 1 | X = 0]$.

b) **Find** both $\text{Var}\, X$ and $\text{Cov}(l, X)$.

c) **Find** the moment generating function of *X*, $M_X(t)$.

9. Random variables X and Y are jointly continuous with pdf

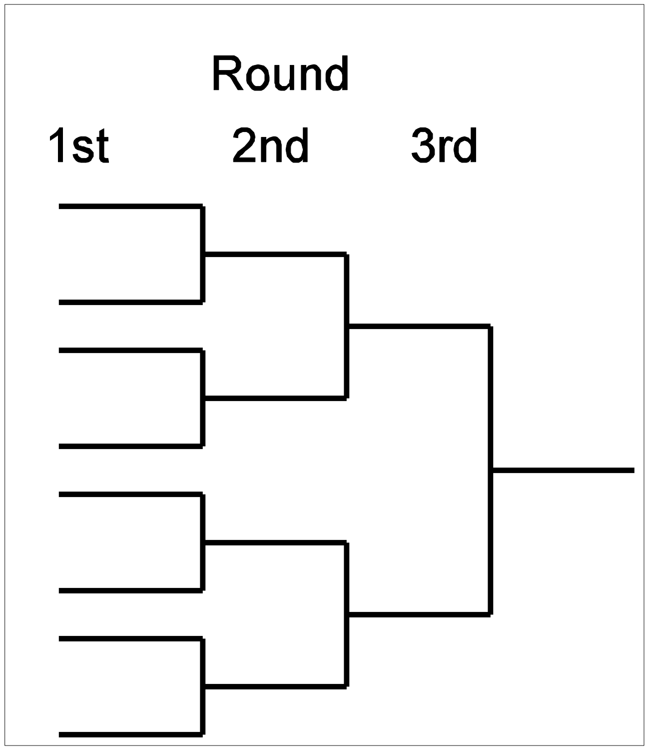
$$f(x,y) = \begin{cases} \frac{3}{2}\left(x^2 + y^2\right) & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) For $0 < x < 1$ and $0 < y < 1$ **find** the value of the (joint) cdf $F(x,y)$.

b) **Find** a marginal pdf for X. (Be careful to specify it completely.)

c) **Are** X and Y independent? **Explain**.

d) **Find** a pdf for $W = X^2$.


Round
1st
2nd
3rd