KEY

Stat 542 Exam 2

November 15, 2001 Prof. Vardeman

1. Below is a table specifying the discrete joint distribution of random variables X and Y.

$y \backslash x$	0	1	2	3	4	5
5	.25	0	0	.01	.03	.02
4	0	0	.01	.03	.02	.03
3	0	.01	.03	.02	.03	.01
2	.01	.03	.02	.03	.01	0
1	.03	.02	.03	.01	0	0
0	.02	.03	.01	0	0	.25

7pts a) Evaluate $P[X+Y>2]$.

$$P[X+Y>2] = 1-P[X+Y \le 2]$$
$$= 1-(.02+.03+.01+.03+.02+.01)$$
$$= .88$$

7pts b) Evaluate $\mathrm{E}[Y|X=4]$.

y	$f(y\|4)$	$y f(y\|4)$
5	3/9	15/9
4	2/9	8/9
3	3/9	9/9
2	1/9	2/9
		34/9

$$E[Y \mid X=4] = \frac{34}{9}$$

7 pts c) Suppose that

$$W = \begin{cases} X+Y & \text{if both } X \text{ and } Y \text{ are odd} \\ 0 & \text{otherwise} \end{cases}$$

Write out a sum that is EW. (You need not do the arithmetic necessary to simplify this.)

$$EW = (1+1)(.02) + (1+3)(.01) + (3+1)(.01)$$
$$+ (3+3)(.02) + (3+5)(.01) + (5+3)(.01) + (5+5)(.02)$$

2. (Moore) Suppose that in appropriate units, the following is true. The length of a standard bar of steel is μ. A copy of the bar is not perfect, and has length L_1 so that $D_1 = L_1 - \mu \sim \mathrm{N}(0,1)$. A copy of this copy has length L_2 so that $D_2 = L_2 - L_1 \sim \mathrm{N}(0,1)$ and D_2 is independent of L_1. A copy of the copy of the copy has length L_3 so that $D_3 = L_3 - L_2 \sim \mathrm{N}(0,1)$ and D_3 is independent of L_1 and L_2.

10 pts a) What is the (joint) distribution of $\boldsymbol{L} = (L_1, L_2, L_3)'$?

$$D = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} \sim MVN_3(0, I) \qquad L = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} D + \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix}$$

$$AIA' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$L \sim MVN_3\left(\begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}\right)$$

5 pts b) Evaluate the correlation between L_1 and L_3.

$$\rho_{13} = \frac{\mathrm{Cov}(L_1, L_3)}{\sqrt{\mathrm{Var}\,L_1 \cdot \mathrm{Var}\,L_3}} = \frac{1}{\sqrt{1 \cdot 3}} = \frac{1}{\sqrt{3}}$$

5 pts c) What is the distribution of $L_3 - \mu$, the error in length of the last bar?

$$L_3 \sim N(\mu, 3) \quad \text{so} \quad L_3 - \mu \sim N(0, 3)$$

3. Suppose that $U \sim$ Uniform $(0,1)$ and that the conditional distribution of $Y|U = u$ is $\mathrm{N}(u,1)$.

7pts a) Evaluate Var$Y$.

$$\mathrm{Var}\,Y = E\,\mathrm{Var}[Y|U] + \mathrm{Var}\,E[Y|U]$$
$$= 1 + \mathrm{Var}\,U$$
$$= 1 + \frac{1}{12} = \frac{13}{12}$$

7 pts b) Evaluate Cov(U, Y).

$$\mathrm{Cov}(U,Y) = EUY - EUEY$$
$$= E(U\,E[Y|U]) - \tfrac{1}{2}\,EE[Y|U]$$
$$= E(U^2) - \tfrac{1}{2}EU$$
$$= EU^2 - \tfrac{1}{4} = \mathrm{Var}\,U = \frac{1}{12}$$

4. Suppose that X and Y are jointly "uniform on the unit circle," i.e. jointly continuous with pdf

$$f(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

7 pts a) For $x \in (-1, 1)$, what is the conditional distribution of $Y|X = x$?

$$f(y|x) = \begin{cases} \text{constant} & y \in (-\sqrt{1-x^2}, \sqrt{1-x^2}) \\ 0 & \text{otherwise} \end{cases}$$

i.e. the conditional dsn of $Y|X=x$ is Uniform on $(-\sqrt{1-x^2}, \sqrt{1-x^2})$

5 pts b) Are X and Y independent? Explain.

X and Y are clearly not independent... the conditional dsn of $Y|X=x$ changes with x

20 pts 5. Find a pdf for $S = \frac{Z}{U}$ where Z and U are independent random variables, $Z \sim \mathrm{N}(0,1)$ and $U \sim$ Uniform $(0,1)$. (Among other possibilities, a transformation will work here.)

$$f(z,u) = \begin{cases} \phi(z) & 0<u<1 \text{ and } z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

Let $h: \begin{cases} S = \frac{Z}{U} \\ T = U \end{cases}$ Then $h^{-1}: \begin{cases} Z = ST \\ U = T \end{cases}$

h is 1-1 from $\{(z,u) \in \mathbb{R}^2 \mid u \in (0,1)\}$ onto $\{(s,t) \in \mathbb{R}^2 \mid t \in (0,1)\}$ and further is cont$\equiv$ly differentiable with

$$J_{h^{-1}}((s,t)) = \begin{vmatrix} t & s \\ 0 & 1 \end{vmatrix} = t$$

Hence $$f(s,t) = \begin{cases} t\,\phi(st) & t \in (0,1),\ s \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

Then
$$f(s) = \int f(s,t)\,dt$$
$$= \int_0^1 \frac{t}{\sqrt{2\pi}} e^{-\frac{s^2t^2}{2}}\,dt$$
$$= \begin{cases} \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-\frac{s^2t^2}{2}}}{s^2} \Bigg|_{t=0}^{1} \right] & s \neq 0 \\ \frac{1}{\sqrt{2\pi}} & s = 0 \end{cases}$$
$$= \begin{cases} \frac{1}{\sqrt{2\pi}\, s^2} \left[1 - e^{-\frac{s^2}{2}} \right] & s \neq 0 \\ \frac{1}{\sqrt{2\pi}} & s = 0 \end{cases}$$

6. The Poisson (λ) moment generating function is

$$M(t) = \exp(\lambda(\exp(t) - 1))$$

7 pts (a) If $X \sim$ Poisson (λ), what is the mgf of $Z = (X - \lambda)/\sqrt{\lambda}$, say $M_Z(t)$?

$Z = \frac{1}{\sqrt{\lambda}} X - \sqrt{\lambda}$ so

$$M_Z(t) = e^{-\sqrt{\lambda}\, t} \, e^{\lambda\left(e^{\frac{t}{\sqrt{\lambda}}} - 1\right)}$$

$$= e^{-\lambda - \sqrt{\lambda}\, t + \lambda e^{\frac{t}{\sqrt{\lambda}}}}$$

6 pts (b) A second order Taylor expansion implies that $\exp(x) = 1 + x + \frac{x^2}{2} + R(x)$ where $|R(x)| \le \frac{\exp(|x|)|x|^3}{6}$. Apply this to the form you found in part (a) and identify a limit for $M_Z(t)$ as $\lambda \to \infty$. (Don't approximate the "outside exponential," only its argument.)

From above

$$M_Z(t) = \exp\left(-\lambda - \sqrt{\lambda}\, t + \lambda\left(1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + R\left(\frac{t}{\sqrt{\lambda}}\right)\right)\right)$$

$$= \exp\left(\frac{t^2}{2} + \lambda R\left(\frac{t}{\sqrt{\lambda}}\right)\right)$$

$$= e^{\frac{t^2}{2}} \cdot \exp\left(\lambda R\left(\frac{t}{\sqrt{\lambda}}\right)\right)$$

Now $\left|\lambda R\left(\frac{t}{\sqrt{\lambda}}\right)\right| \le \lambda \left|\frac{t}{\sqrt{\lambda}}\right|^3 \frac{1}{6} \exp\left(\left|\frac{t}{\sqrt{\lambda}}\right|\right)$

Since t is fixed here, this has limit $0 \cdot \exp(0) = 0$

So $M_Z(t) \longrightarrow e^{\frac{t^2}{2}} \leftarrow$ the $N(0,1)$ mgf evaluated at t