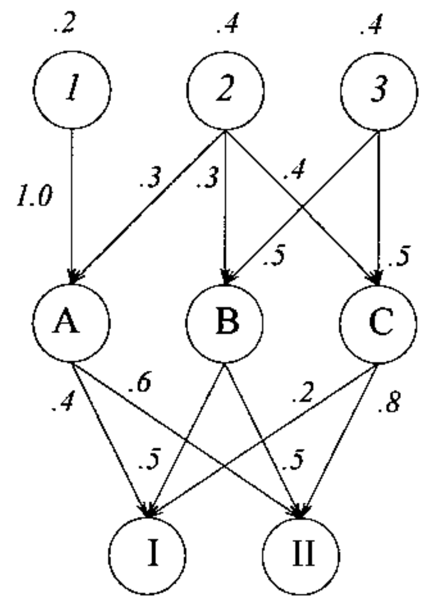
KEY

Stat 542 Exam I

October 11, 2001 Prof. Vardeman

1. A rat enters a maze at one of the nodes 1, 2 and 3, proceeds through one of nodes A, B and C, to end at one of nodes I and II. Below is a schematic of the maze with (conditional) probabilities marked on it indicating probabilities of taking various next paths, given the rat reaches a particular node. (The values at the top of the diagram indicate the probabilities of beginning at nodes 1, 2 and 3.)



7pts a) Write out (but you need not evaluate) a numerical expression for the probability that a rat ends at node II.

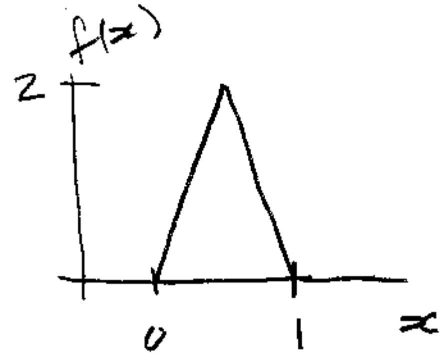
$$P[II] = P[1A\,II] + P[2A\,II] + P[2B\,II] + P[2C\,II] + P[3B\,II] + P[3C\,II]$$
$$= (.2)(1)(.6) + (.4)(.3)(.6) + (.4)(.3)(.5) + (.4)(.4)(.8)$$
$$+ (.4)(.5)(.5) + (.4)(.5)(.8)$$

8pts b) Write out (but you need not evaluate) a numerical expression for the conditional probability that a rat began at node 2, given that it ends at node II.

$$P[2 \mid II] = \frac{P[2 \text{ and } II]}{P[II]}$$
$$= \frac{(.4)(.3)(.6) + (.4)(.3)(.5) + (.4)(.4)(.8)}{\text{answer to a)}}$$

2. A random variable X has pdf

$$f(x) = \begin{cases} 4x & 0 < x < \frac{1}{2} \\ 4(1-x) & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



Carefully determine the following:

6pts a) EX

$EX = \frac{1}{2}$ by symmetry

6pts b) Var$X$

$$EX^2 = \int_0^{\frac{1}{2}} x^2(4x)\,dx + \int_{\frac{1}{2}}^1 4x^2(1-x)\,dx$$

$$= x^4\Big|_0^{\frac{1}{2}} + \frac{4}{3}x^3\Big|_{\frac{1}{2}}^1 - x^4\Big|_{\frac{1}{2}}^1$$

$$= \frac{1}{16} + \frac{4}{3} - \frac{1}{6} - 1 + \frac{1}{16} = \frac{7}{24}$$

So $\text{Var}\,X = \frac{7}{24} - \left(\frac{1}{2}\right)^2 = \frac{1}{24}$

6pts c) $F(x)$

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \int_0^x 4y\,dy & \text{if } x \in (0, \frac{1}{2}) \\ \frac{1}{2} + \int_{\frac{1}{2}}^x 4(1-y)\,dy & \text{if } x \in [\frac{1}{2}, 1) \\ 1 & \text{if } x \geq 1 \end{cases} = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x^2 & \text{if } x \in (0, \frac{1}{2}) \\ -2x^2 + 4x - 1 & \text{if } x \in [\frac{1}{2}, 1] \\ 1 & \text{if } x \geq 1 \end{cases}$$

6pts d) A pdf for the random variable $Y = -\ln(x)$

$y = -\ln(x)$ maps $(0,1)$ onto $(0,\infty)$ so the pdf for Y is 0 unless $y \in (0,\infty)$. Note also that $x = e^{-y}$ so that $\frac{dx}{dy} = -e^{-y}$. Then for $y > 0$

$$f_Y(y) = f(e^{-y})\,|-e^{-y}|$$

i.e.
$$f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 4e^{-2y} & \text{if } 0 < e^{-y} < \frac{1}{2} \text{ i.e. if } y > \ln 2 \\ 4(1-e^{-y})e^{-y} & \text{if } y \in (0, \ln 2) \end{cases}$$

3. A famous (and incredibly popular) statistician has 8 shirts, 5 of which are blue. He wears a clean one each day of the week, does laundry on the weekends and begins each Monday morning with a closet full of 8 clean shirts. He teaches a course that meets 4 days per week.

Suppose that this person actually chooses which shirt he wears on a given day at random from those remaining in his closet.

7pts a) What is the probability that in a particular week of 4 lectures, his students see only blue shirts?

$$\frac{\binom{5}{4}\binom{3}{0}}{\binom{8}{4}} = \frac{1}{14}$$

5pts b) What is the probability that in a particular week of 4 lectures, the students see their 2nd blue shirt on day 4?

$$P[\text{2nd blue on day 4}] = P[\text{blue on day 4} \mid \text{1 blue in 1st 3 days}] \cdot P[\text{1 blue in 1st 3 days}]$$
$$= \frac{4}{5} \cdot \frac{\binom{5}{1}\binom{3}{2}}{\binom{8}{3}} = \frac{12}{56} = \frac{3}{14}$$

6pts c) What is the probability that during the first 6 weeks of class, the students see only blue shirts?

$$\left(\frac{1}{14}\right)^6$$

5pts 4. Suppose that $U \sim$ Uniform $(0, 1)$ and that F is a continuous, strictly increasing cdf (so that F has an inverse, F^{-1}). What is the distribution of $Y = F^{-1}(U)$? Argue carefully for your answer.

$$P[Y \le y] = P[F^{-1}(U) \le y] = P[F(F^{-1}(U)) \le F(y)]$$
$$= P[U \le F(y)]$$
$$= F(y)$$

i.e. $Y \sim F$

8pts 5. Suppose that $X \sim \text{Poisson}(\lambda)$. Evaluate $\text{E}(\frac{1}{X+1})$.

$$E\frac{1}{X+1} = \sum_{x=0}^{\infty} \frac{1}{x+1} \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{\lambda}\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$$

$$= \frac{1}{\lambda}\sum_{y=1}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}$$

$$= \frac{1}{\lambda}\left(1-e^{-\lambda}\right)$$

6. Suppose that $Z \sim \text{Normal}\ (0, 1)$. I know something about the realized value of Z and tell you that $Z > -1$.

6pts a) What do you propose to use as a cdf for Z, *conditional on the information that* $Z > -1$? (What is $P[Z \leq z | Z > -1]$?)

$$P[Z \leq z \mid Z > -1] = \begin{cases} 0 & \text{if } z \leq -1 \\ \dfrac{\Phi(z) - \Phi(-1)}{1-\Phi(-1)} & \text{if } z > -1 \end{cases}$$

4pts b) Write out an explicit expression for a sensible "mean value of Z conditioned on $Z > -1$" (but don't bother to evaluate this).

A sensible conditional pdf is $\frac{d}{dz}$ (above) i.e.

$$f(z \mid Z > -1) = \begin{cases} 0 & \text{if } z < -1 \\ \dfrac{\phi(z)}{1-\Phi(z)} & \text{if } z > -1 \end{cases}$$

So a sensible conditional mean is

$$\int_{-1}^{\infty} \frac{z\,\phi(z)}{1-\Phi(z)}\,dz$$

10pts 7. Find a expression for $\mathrm{Var}(3X(X-1)+5)$ in terms of moments of X, $\mathrm{E}X^r$, $r = 1, 2, 3, 4$.

$$
\begin{aligned}
\mathrm{Var}\left(3X(X-1)+5\right) &= 9\,\mathrm{Var}\, X(X-1) \\
&= 9\left[E\left[X(X-1)\right]^2 - \left[EX(X-1)\right]^2\right] \\
&= 9\left[E\left(X^4 - 2X^3 + X^2\right) - \left(EX^2 - EX\right)^2\right] \\
&= 9\Big[EX^4 - 2EX^3 + EX^2 - \left(EX^2\right)^2 + 2(EX)(EX^2) \\
&\qquad - (EX)^2\Big]
\end{aligned}
$$