Graduate Lectures and Problems in Quality Control and Engineering Statistics: Theory and Methods

To Accompany

Statistical Quality Assurance Methods for Engineers

by

Vardeman and Jobe

Stephen B. Vardeman

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Chapter 6

Problems

1 Measurement and Statistics

1.1. Suppose that a sample variance $s^2$ is based on a sample of size $n$ from a normal distribution. One might consider estimating $\sigma$ using $s$ or $s/c_4(n)$, or even some other multiple of $s$.

(a) Since $c_4(n) < 1$, the second of these estimators has a larger variance than the first. But the second is unbiased (has expected value $\sigma$) while the first is not. Which has the smaller mean squared error, $E(\hat{\sigma} - \sigma)^2$? Note that (as is standard in statistical theory), $E(\hat{\sigma} - \sigma)^2 = \text{Var} \hat{\sigma} + (E \hat{\sigma} - \sigma)^2$. (Mean squared error is variance plus squared bias.)

(b) What is an optimal (in terms of minimum mean squared error) multiple of $s$ to use in estimating $\sigma$?

1.2. How do $R/d_2(n)$ and $s/c_4(n)$ compare (in terms of mean squared error) as estimators of $\sigma$? (The assumption here is that they are both based on a sample from a normal distribution. See Problem 1.1 for a definition of mean squared error.)

1.3. Suppose that sample variances $s^2_i$, $i = 1, 2, \ldots, r$ are based on independent samples of size $m$ from normal distributions with a common standard deviation, $\sigma$. A common SQC-inspired estimator of $\sigma$ is $\bar{s}/c_4(m)$. Another possibility is

$$s_{\text{pooled}} = \sqrt{\left( \frac{s_1^2 + \cdots + s_r^2}{r} \right)}$$
or
\[ \hat{\sigma} = s_{\text{pooled}} / c_4((m - 1)r + 1) . \]

Standard distribution theory says that \( r(m - 1)s^2_{\text{pooled}} / \sigma^2 \) has a \( \chi^2 \) distribution with \( r(m - 1) \) degrees of freedom.

(a) Compare \( \hat{s}/c_4(m) \), \( s_{\text{pooled}} \) and \( \hat{\sigma} \) in terms of mean squared error.

(b) What is an optimal multiple of \( s_{\text{pooled}} \) (in terms of mean squared error) to use in estimating \( \sigma \)?

(Note: See Vardeman (1999 IIE Transactions) for a complete treatment of the issues raised in Problems 1.1 through 1.3.)

1.4. Set up a double integral that gives the probability that the sample range of \( n \) standard normal random variables is between .5 and 2.0. How is this probability related to the probability that the sample range of \( n \) iid normal \( (\mu, \sigma^2) \) random variables is between \( .5\sigma \) and \( 2.0\sigma \)?

1.5. It is often helpful to state “standard errors” (estimated standard deviations) corresponding to point estimates of quantities of interest. In a context where a standard deviation, \( \sigma \), is to be estimated by \( R/\sqrt{d_2(n)} \) based on \( r \) samples of size \( n \), what is a reasonable standard error to announce? (Be sure that your answer is computable from sample data, i.e. doesn’t involve any unknown process parameters.)

1.6. Consider the paper weight data in Problem (2.12) of V&J. Assume that the 2-way random effects model is appropriate and do the following.

(a) Compute the \( \bar{y}_{ij} \), \( s_{ij} \) and \( R_{ij} \) for all \( I \times J = 2 \times 5 = 10 \) Piece\times Operator combinations. Then compute both row ranges of means \( \Delta_i \) and row sample variances of means \( s_i^2 \).

(b) Find both range-based and sample variance-based point estimates of the repeatability standard deviation, \( \sigma \).

(c) Find both range-based and sample variance-based point estimates of the reproducibility standard deviation \( \sigma_{\text{reproducibility}} = \sqrt{\sigma^2_\beta + \sigma^2_{\alpha\beta}} \).

(d) Get a statistical package to give you the 2-way ANOVA table for these data. Verify that \( s^2_{\text{pooled}} = MSE \) and that your sample variance-based estimate of \( \sigma_{\text{reproducibility}} \) from part (c) is
\[ \sqrt{\max \left( 0, \frac{1}{mI} MSB + \frac{I - 1}{mI} MSAB - \frac{1}{m} MSE \right)} . \]
1. MEASUREMENT AND STATISTICS

(e) Find a 90% two-sided confidence interval for the parameter $\sigma$.

(f) Use the material in §1.5 and give an approximate 90% two-sided confidence interval for $\sigma_{\text{reproducibility}}$.

(g) Find a linear combination of the mean squares from (d) whose expected value is $\sigma^2_{\text{overall}} = \sigma^2_{\text{reproducibility}} + \sigma^2$. All the coefficients in your linear combination will be positive. In this case, you may use the next to last paragraph of §1.5 to come up with an approximate 90% two-sided confidence interval for $\sigma_{\text{overall}}$. Do so.

(h) The problem from which the paper weight data are drawn indicates that specifications of approximately $\pm 4\,\text{g/m}^2$ are common for paper of the type used in this gage study. These translate to specifications of about $\pm 0.16\,\text{g}$ for pieces of paper of the size used here. Use these specifications and your answer to part (g) to make an approximate 90% confidence interval for the gage capability ratio

$$GCR = \frac{6\sigma_{\text{overall}}}{(U - L)}.$$  

Used in the way it was in this study, does the scale seem adequate to check conformance to such specifications?

(i) Give (any sensible) point estimates of the fractions of the overall measurement variance attributable to repeatability and to reproducibility.

1.7. In a particular (real) thorium detection problem, measurement variation for a particular (spectral absorption) instrument was thought to be about $\sigma_{\text{measurement}} = 0.002$ instrument units. (Division of a measurement expressed in instrument units by 58.2 gave values in g/l.) Suppose that in an environmental study, a field sample is to be measured once (producing $y_{\text{new}}$) on this instrument and the result is to be compared to a (contemporaneous) measurement of a lab “blank” (producing $y_{\text{old}}$). If the field reading exceeds the blank reading by too much, there will be a declaration that there is a detectable excess amount of thorium present.

(a) Assuming that measurements are normal, find a critical value $L_c$ so that the lab will run no more than a 5% chance of a “false positive” result.

(b) Based on your answer to (a), what is a “lower limit of detection,” $L_d$, for a 90% probability ($\gamma$) of correctly detecting excess thorium? What, by the way, is this limit in terms of g/l?
1.8. Below are 4 hypothetical samples of size \( n = 3 \). A little calculation shows that ignoring the fact that there are 4 samples and simply computing "s" based on 12 observations will produce a “standard deviation” much larger than \( s_{pooled} \). Why is this?

\[ 3,6,5 \quad 4,3,1 \quad 8,9,6 \quad 2,1,4 \]

1.9. In applying ANOVA methods to gage R&R studies, one often uses linear combinations of independent mean squares as estimators of their expected values. Section 1.5 of these notes shows it is possible to also produce standard errors (estimated standard deviations) for these linear combinations. Suppose that \( MS_1, MS_2, \ldots, MS_k \) are independent random variables, \( \sum_{i=1}^{k} MS_i \sim \chi^2_{\nu_i} \). Consider the random variable

\[ U = c_1 MS_1 + c_2 MS_2 + \cdots + c_k MS_k \]

(a) Find the standard deviation of \( U \).

(b) Your expression from (a) should involve the means \( EMS_i \), that in applications will be unknown. Propose a sensible (data-based) estimator of the standard deviation of \( U \) that does not involve these quantities.

(c) Apply your result from (b) to give a sensible standard error for the ANOVA-based estimators of \( \sigma^2 \), \( \sigma^2_{reproducibility} \) and \( \sigma^2_{overall} \).

1.10. Section 1.7 of the notes presents “rounded data” likelihood methods for normal data with the 2 parameters \( \mu \) and \( \sigma \). The same kind of thing can be done for other families of distributions (which can have other numbers of parameters). For example, the exponential distributions with means \( \theta^{-1} \) can be used. (Here there is the single parameter \( \theta \).) These exponential distributions have cdf’s

\[ F_{\theta}(x) = \begin{cases} 1 - \exp(-\theta x) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \]

Below is a frequency table for twenty exponential observations that have been rounded to the nearest integer.

<table>
<thead>
<tr>
<th>rounded value</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>frequency</td>
<td>7</td>
<td>8</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Write out an expression for the appropriate “rounded data log likelihood function” for this problem,

\[ L(\theta) = \ln L(data|\theta) \].
(You should be slightly careful here. Exponential random variables only take values in the interval \((0, \infty)\).)

(b) Make a plot of \(\mathcal{L}(\theta)\). Use it and identify the maximum likelihood estimate of \(\theta\) based on the rounded data.

(c) Use the plot from (b) and make an approximate 90% confidence interval for \(\theta\). (The appropriate \(\chi^2\) value has 1 associated degree of freedom.)

1.11. Below are values of a critical dimension (in .0001 inch above nominal) measured on hourly samples of size \(n = 5\) precision metal parts taken from the output of a CNC (computer numerically controlled) lathe.

<table>
<thead>
<tr>
<th>sample</th>
<th>measurements</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>4,1,0,1,0</td>
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<tr>
<td>5</td>
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<tr>
<td>6</td>
<td>2, 2, 2, 1, 2</td>
</tr>
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<td>0,0,0,2,0</td>
</tr>
<tr>
<td>8</td>
<td>1,1,2,0,2</td>
</tr>
</tbody>
</table>

(a) Compute for each of these samples the “raw” sample standard deviation (ignoring rounding) and the “Sheppard’s correction” standard deviation that is appropriate for integer rounded data. How do these compare for the eight samples above?

(b) For each of the samples that have a range of at least 2, use the CONEST program to find “rounded normal data” maximum likelihood estimates of the normal parameters \(\mu\) and \(\sigma\). The program as written accepts observations \(\geq 1\), so you will need to add an integer to each element of some of the samples above before doing calculation with the program. (I don’t remember, but you may not be able to input a standard deviation of exactly 0 either.) How do the maximum likelihood estimates of \(\mu\) compare to \(\bar{x}\) values? How do the maximum likelihood estimates of \(\sigma\) compare to both the raw standard deviations and to the results of applying “Sheppard’s correction”?

(c) Consider sample #2. Make 95% and 90% confidence intervals for both \(\mu\) and \(\sigma\) using the work of Johnson Lee.

(d) Consider sample #1. Use the CONEST program to get a few approximate values for \(\mathcal{L}^*(\mu)\) and some approximate values for \(\mathcal{L}^{**}(\sigma)\). (For example, look at a contour plot of \(\mathcal{L}\) over a narrow range of means near \(\mu\) to get an approximate value for \(\mathcal{L}^*(\mu)\).) Sketch \(\mathcal{L}^*(\mu)\) and \(\mathcal{L}^{**}(\sigma)\) and use your sketches and Lee’s tables to produce 95% confidence intervals for \(\mu\) and \(\sigma\).

(e) What 95% confidence intervals for \(\mu\) and \(\sigma\) would result from a 9th sample, \(\{2, 2, 2, 2, 2\}\)?
1.12. A single operator measures a single widget diameter 15 times and obtains a range of \( R = 3 \times 10^{-4} \) inches. Then this person measures the diameters of 12 different widgets once each and obtains a range of \( R = 8 \times 10^{-4} \) inches. Give an estimated standard deviation of widget diameters (not including measurement error).

1.13. Cylinders of (outside) diameter \( O \) must fit in ring bearings of (inside) diameter \( I \), producing clearance \( C = I - O \). We would like to have some idea of the variability in actual clearances that will be obtained by “random assembly” of cylinders produced on one production line with ring bearings produced on another. The gages used to measure \( I \) and \( O \) are (naturally enough) different.

In a study using a single gage to measure outside diameters of cylinders, \( n_O = 10 \) different cylinders were measured once each, producing a sample standard deviation \( s_O = .001 \) inch. In a subsequent study, this same gage was used to measure the outside diameter of an additional cylinder \( m_O = 5 \) times, producing a sample standard deviation \( s_{O gage} = .0005 \) inch.

In a study using a single gage to measure inside diameters of ring bearings, \( n_I = 20 \) different inside diameters were measured once each, producing a sample standard deviation \( s_I = .003 \) inch. In a subsequent study, this same gage was used to measure the inside diameter of another ring bearing \( m_I = 10 \) times, producing a sample standard deviation \( s_{I gage} = .001 \) inch.

(a) Give a sensible (point) estimate of the standard deviation of \( C \) produced under random assembly.

(b) Find a sensible standard error for your estimate in (a).

2 Process Monitoring

Methods

2.1. Consider the following hypothetical situation. A “variables” process monitoring scheme is to be set up for a production line, and two different measuring devices are available for data gathering purposes. Device A produces precise and expensive measurements and device B produces less precise and less expensive measurements. Let \( \sigma_{\text{measurement}} \) for the two devices be respectively \( \sigma_A \) and \( \sigma_B \), and suppose that the target for a particular critical diameter for widgets produced on the line is 200.0.
2. PROCESS MONITORING

(a) A single widget produced on the line is measured $n = 10$ times with each device and $R_A = 2.0$ and $R_B = 5.0$. Give estimates of $\sigma_A$ and $\sigma_B$.

(b) Explain why it would not be appropriate to use one of your estimates from (a) as a “$\sigma$” for setting up an $\bar{x}$ and $R$ chart pair for monitoring the process based on measurements from one of the devices.

Using device A, 10 consecutive widgets produced on the line (under presumably stable conditions) have (single) measurements with $R = 8.0$.

(c) Set up reasonable control limits for both $\bar{x}$ and $R$ for the future monitoring of the process based on samples of size $n = 10$ and measurements from device A.

(d) Combining the information above about the A measurements on 10 consecutive widgets with your answer to (a), under a model that says

\[
\text{observed diameter} = \text{real diameter} + \text{measurement error}
\]

where “real diameter” and “measurement error” are independent, give an estimate of the standard deviation of the real diameters. (See the discussion around page 19 of V&J.)

(e) Based on your answers to parts (a) and (d), set up reasonable control limits for both $\bar{x}$ and $R$ for the future monitoring of the process based on samples of size $n = 5$ and measurements from the cheaper device, device B.

2.2. The following are some data taken from a larger set in Statistical Quality Control by Grant and Leavenworth, giving the drained weights (in ounces) of contents of size No. $2\frac{1}{2}$ cans of standard grade tomatoes in puree. 20 samples of three cans taken from a canning process at regular intervals are represented.
Suppose that standard values for the process mean and standard deviation of drained weights ($\mu$ and $\sigma$) in this canning plant are 21.0 oz and 1.0 oz respectively. Make and interpret standards given $\bar{x}$ and $R$ charts based on these samples. What do these charts indicate about the behavior of the filling process over the time period represented by these data?

As an alternative to the standards given range chart made in part (a), make a standards given $s$ chart based on the 20 samples. How does its appearance compare to that of the $R$ chart?

Now suppose that no standard values for $\mu$ and $\sigma$ have been provided.

Find one estimate of $\sigma$ for the filling process based on the average of the 20 sample ranges, $\bar{R}$, and another based on the average of 20 sample standard deviations, $\bar{s}$.

Use $\bar{R}$ and your estimate of $\sigma$ based on $\bar{R}$ and make retrospective control charts for $\bar{x}$ and $R$. What do these indicate about the stability of the filling process over the time period represented by these data?

Use $\bar{s}$ and your estimate of $\sigma$ based on $\bar{s}$ and make retrospective control charts for $\bar{x}$ and $s$. How do these compare in appearance to the retrospective charts for process mean and variability made in part (d)?

The accompanying data are some taken from Statistical Quality Control Methods by I.W. Burr, giving the numbers of beverage cans found to be defective in periodic samples of 312 cans at a bottling facility.
2. PROCESS MONITORING

<table>
<thead>
<tr>
<th>Sample</th>
<th>Defectives</th>
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<td>10</td>
<td>6</td>
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<table>
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</thead>
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<td>19</td>
<td>8</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
</tr>
</tbody>
</table>

(a) Suppose that company standards are that on average \( p = 0.02 \) of the cans are defective. Use this value and make a standards given \( p \) chart based on the data above. Does it appear that the process fraction defective was stable at the \( p = 0.02 \) value over the period represented by these data?

(b) Make a retrospective \( p \) chart for these data. What is indicated by this chart about the stability of the canning process?

2.4. Modern business pressures are making standards for fractions nonconforming in the range of \( 10^{-4} \) to \( 10^{-6} \) not uncommon.

(a) What are standards given \( 3\sigma \) control limits for a \( p \) chart with standard fraction nonconforming \( 10^{-4} \) and sample size 100? What is the all-OK ARL for this scheme?

(b) If \( p \) becomes twice the standard value (of \( 10^{-4} \)), what is the ARL for the scheme from (a)? (Use your answer to (a) and the binomial distribution for \( n = 100 \) and \( p = 2 \times 10^{-4} \).)

(c) What do (a) and (b) suggest about the feasibility of doing process monitoring for very small fractions defective based on attributes data?

2.5. Suppose that a dimension of parts produced on a certain machine over a short period can be thought of as normally distributed with some mean \( \mu \) and standard deviation \( \sigma = 0.005 \) inch. Suppose further, that values of this dimension more than .0098 inch from the 1.000 inch nominal value are considered nonconforming. Finally, suppose that hourly samples of 10 of these parts are to be taken.
(a) If $\mu$ is exactly on target (i.e. $\mu = 1.000$ inch) about what fraction of parts will be nonconforming? Is it possible for the fraction nonconforming to ever be any less than this figure?

(b) One could use a $p$ chart based on $n = 10$ to monitor process performance in this situation. What would be standards given 3 sigma control limits for the $p$ chart, using your answer from part (a) as the standard value of $p$?

(c) What is the probability that a particular sample of $n = 10$ parts will produce an out-of-control signal on the chart from (b) if $\mu$ remains at its standard value of $\mu = 1.000$ inch? How does this compare to the same probability for a 3 sigma $\bar{x}$ chart for an $n = 10$ setup with a center line at 1.000? (For the $p$ chart, use a binomial probability calculation. For the $\bar{x}$ chart, use the facts that $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. ) What are the ARLs of the monitoring schemes under these conditions?

(d) Compare the probability that a particular sample of $n = 10$ parts will produce an out-of-control signal on the $p$ chart from (b) to the probability that the sample will produce an out of control signal on the ($n = 10$) 3 sigma $\bar{x}$ chart first mentioned in (c), supposing that in fact $\mu = 1.005$ inch. What are the ARLs of the monitoring schemes under these conditions? What moral is told by your calculations here and in part (c)?

2.6. The article “High Tech, High Touch,” by J. Ryan, that appeared in Quality Progress in 1987 discusses the quality enhancement processes used by Martin Marietta in the production of the space shuttle external (liquid oxygen) fuel tanks. It includes a graph giving counts of major hardware nonconformities for each of 41 tanks produced. The accompanying data are approximate counts read from that graph for the last 35 tanks. (The first six tanks were of a different design than the others and are thus not included here.)
2. PROCESS MONITORING

2.7. Boulaevskaya, Fair and Seniva did a study of “defect detection rates” for the visual inspection of some glass vials. Vials known to be visually identifiable as defective were marked with invisible ink, placed among other vials, and run through a visual inspection process at 10 different time periods. The numbers of marked defective vials that were detected/captured, the numbers placed into the inspection process, and the corresponding ratios for the 10 periods are below.

<table>
<thead>
<tr>
<th>Tank</th>
<th>Nonconformities</th>
<th>Tank</th>
<th>Nonconformities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>537</td>
<td>19</td>
<td>157</td>
</tr>
<tr>
<td>2</td>
<td>463</td>
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</tbody>
</table>

(a) Make a retrospective $c$ chart for these data. Is there evidence of real quality improvement in this series of counts of nonconformities? Explain.

(b) Consider only the last 17 tanks represented above. Does it appear that quality was stable over the production period represented by these tanks? (Make another retrospective $c$ chart.)

(c) It is possible that some of the figures read from the graph in the original article may differ from the real figures by as much as, say, 15 nonconformities. Would this measurement error account for the apparent lack of stability you found in (a) or (b) above? Explain.
(Overall, 91 of the 225 marked vials placed into the inspection process were detected/captured.)

(a) Carefully investigate (and say clearly) whether there is evidence in these data of instability in the defect detection rate.

(b) $91/225 = 0.404$. Do you think that the company these students worked with was likely satisfied with the 40.4% detection rate? What, if anything, does your answer here have to do with the analysis in (a)?

2.8. **(Narrow Limit Gaging)** Parametric probability model assumptions can sometimes be used to advantage even where one is ultimately going to generate and use attributes data. Consider a situation where process standards are that widget diameters are to be normally distributed with mean $\mu = 5$ and standard deviation $\sigma = 1$. Engineering specifications on these diameters are $5 \pm 3$.

As a process monitoring device, samples of $n = 100$ of these widgets are going to be checked with a go/no-go gage, and $X =$ the number of diameters in a sample failing to pass the gaging test will be counted and plotted on an $np$ chart. The design of the go/no-go gage is up to you to choose. You may design it to pass parts with diameters in any interval $(a, b)$ of your choosing.

(a) One natural choice of $(a, b)$ is according to the engineering specifications, i.e. as $(2, 8)$. With this choice of go/no-go gage, a $3\sigma$ control chart for $X$ signals if $X \geq 2$. Find the all-OK ARL for this scheme with this gage.

(b) One might, however, choose $(a, b)$ in other ways besides according to the engineering specifications, e.g. as $(5 - \delta, 5 + \delta)$ for some $\delta$ other than 3. Show that the choice of $\delta = 2.71$ and a control chart that signals if $X \geq 3$ will have about the same all-OK ARL as the scheme from (a).

(c) Compare the schemes from (a) and (b) supposing that diameters are in fact normally distributed with mean $\mu = 6$ and standard deviation $\sigma = 1$.

2.9. A one-sided upper CUSUM scheme is used to monitor $Q =$ the number of defectives in samples of size $n = 400$. 
2. PROCESS MONITORING

Suppose that one uses \( k_1 = 8 \) and \( h_1 = 10 \). Use the normal approximation to the binomial distribution to obtain an approximate ARL for this scheme if \( p = .025 \).

2.10. Consider the monitoring of a process that we will assume produces normally distributed observations \( X \) with standard deviations \( \sigma = .04 \).

(a) Set up both a two-sided CUSUM scheme and a EWMA scheme for monitoring the process \( (Q = X) \), using a target value of .13 and a desired all-OK ARL of roughly 370, if quickest possible detection of a change in mean of size \( \Delta = .02 \) is desired.

(b) Plot on the same set of axes, the logarithms of the ARLs for your charts from (a) as functions of \( \mu \), the real mean of observations being CUSUMed or EWMAed. Also plot on this same set of axes the logarithms of ARLs for a standard 3\( \sigma \) Shewhart Chart for individuals. Comment upon how the 3 ARL curves compare.

2.11. Shear strengths of spot welds made by a certain robot are approximately normal with a short term variability described by \( \sigma = 60 \) lbs. The strengths in samples of \( n \) of these welds are going to be obtained and \( \bar{x} \) values CUSUMed.

(a) Give a reference value \( k_2 \), sample size \( n \) and a decision interval \( h_2 \) so that a one-sided (lower) CUSUM scheme for the \( \bar{x} \)'s will have an ARL of about 370 if \( \mu = 800 \) lbs and an ARL of about 5 if \( \mu = 750 \) lbs.

(b) Find a sample size and a lower Shewhart control limit for \( \bar{x} \), say \#, so that if \( \mu = 800 \) lbs, there will be about 370 samples taken before an \( \bar{x} \) will plot below \#, and if \( \mu = 750 \) there will be on average about 5 samples taken before an \( \bar{x} \) will plot below \#.

2.12. You have data on the efficiency of a continuous chemical production process. The efficiency is supposed to be about 45\%, and you will use a CUSUM scheme to monitor the efficiency. Efficiency is computed once per shift, but from much past data, you know that \( \sigma \approx .7\% \).

(a) If you wish quickest possible detection of a shift of .7\% (one standard deviation) in mean efficiency, design a two-sided CUSUM scheme for this situation with an all-OK ARL of about 500.
(b) Apply your procedure from (a) to the data below. Are any alarms signaled?

<table>
<thead>
<tr>
<th>Shift</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45.7</td>
</tr>
<tr>
<td>2</td>
<td>44.6</td>
</tr>
<tr>
<td>3</td>
<td>45.0</td>
</tr>
<tr>
<td>4</td>
<td>44.4</td>
</tr>
<tr>
<td>5</td>
<td>44.4</td>
</tr>
<tr>
<td>6</td>
<td>44.2</td>
</tr>
<tr>
<td>7</td>
<td>46.1</td>
</tr>
<tr>
<td>8</td>
<td>44.6</td>
</tr>
<tr>
<td>9</td>
<td>45.7</td>
</tr>
<tr>
<td>10</td>
<td>44.4</td>
</tr>
</tbody>
</table>

(c) Make a plot of “raw” CUSUMs using a reference value of 45%. From your plot, when do you think that the mean efficiency shifted away from 45%?

(d) What are the all-OK and “μ = 45.7%” ARLs if one employs your procedure from (a) modified by giving both the high and low side charts “head starts” of \( u = v = h_1/2 = h_2/2 \)?

(e) Repeat part (a) using a EWMA scheme rather than a CUSUM scheme.

(f) Apply your procedure from (e) to the data. Are any alarms signaled? Plot your EWMA values. Based on this plot, when do you think that the mean efficiency shifted away from 45%?

2.13. Consider the problem of designing a EWMA control chart for \( \bar{x} \)’s, where in addition to choosing chart parameters one gets to choose the sample size, \( n \). In such a case, one can choose monitoring parameters to produce both a desired (large) on-target ARL and a desired (small) off-target ARL \( \delta \) units away from the target.

Suppose, for example, that a process standard deviation is \( \sigma = 1 \) and one wishes to design for an ARL of 370 if the process mean, \( \mu \), is on target, and an ARL of no more than 5.0 if \( \mu \) is off target by as much as \( \delta = 1.0 \). Using \( \sigma_Q = \sigma/\sqrt{n} \) and \( shift = \delta/\sigma_Q \) and reading from one of the graphs in Crowder’s 1989 *JQT* paper, values of \( \lambda^{opt} \) for detecting a change in process mean of this size using EWMAs of \( \bar{x} \)’s are approximately as below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda^{opt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.14</td>
</tr>
<tr>
<td>2</td>
<td>.08</td>
</tr>
<tr>
<td>3</td>
<td>.06</td>
</tr>
<tr>
<td>4</td>
<td>.05</td>
</tr>
<tr>
<td>5</td>
<td>.05</td>
</tr>
<tr>
<td>6</td>
<td>.04</td>
</tr>
<tr>
<td>7</td>
<td>.04</td>
</tr>
<tr>
<td>8</td>
<td>.03</td>
</tr>
</tbody>
</table>

Use Crowder’s EWMA ARL program (and some trial and error) to find values of \( K \) that when used with the \( \lambda \)’s above will produce an on-target
ARL of 370. Then determine how large \( n \) must then be in order to meet the 370 and 5.0 ARL requirements. How does this compare to what Table 4.8 says is needed for a two-sided CUSUM to meet the same criteria?

2.14. Consider a combination of high and low side decision interval CUSUM schemes with \( h_1 = h_2 = 2.5, u = 1, v = -1, k_1 = .5 \) and \( k_2 = -.5 \). Suppose that \( Q \)'s are iid normal variables with \( \sigma_Q = 1.0 \). Find the ARLs for the combined scheme if \( \mu_Q = 0 \) and then if \( \mu_Q = 1.0 \). (You will need to use Gan’s CUSUM ARL program and Yashchin’s expression for combining high and low side ARLs.)

2.15. Set up two different \( X/MR \) monitoring chart pairs for normal variables \( Q \), in the case where the standards are \( \mu_Q = 5 \) and \( \sigma_Q = 1.715 \) and the all-OK ARL desired is 250. For these combinations, what ARLs are relevant if in fact \( \mu_Q = 5.5 \) and \( \sigma_Q = 2.00 \)? (Run Crowder’s \( X/MR \) ARL program to get these with minimum interpolation.)

2.16. If one has discrete or rounded data and insists on using \( \bar{x} \) and/or \( R \) charts, §1.7.1 shows how these may be based on the exact all-OK distributions of \( \bar{x} \) and/or \( R \) (and not on normal theory control limits). Suppose that measurements arise from integer rounding of normal random variables with \( \mu = 2.25 \) and \( \sigma = .5 \) (so that essentially only values 1, 2, 3 and 4 are ever seen). Compute the four probabilities corresponding to these rounded values (and “fudge” them slightly so that they total to 1.00). Then, for \( n = 4 \) compute the probability distributions of \( \bar{x} \) and \( R \) based on iid observations from this distribution. Then run Karen (Jensen) Hulting’s DIST program and compare your answers to what her program produces.

2.17. Suppose that standard values of process parameters are \( \mu = 17 \) and \( \sigma = 2.4 \).

(a) Using sample means \( \bar{x} \) based on samples of size \( n = 4 \), design both a combined high and low side CUSUM scheme (with 0 head starts) and a EWMA scheme to have an all-OK ARL of 370 and quickest possible detection of a shift in process of mean of size .6.

(b) If, in fact, the process mean is \( \mu = 17.5 \) and the process standard deviation is \( \sigma = 3.0 \), show how you would find the ARL associated with your schemes from (a). (You don’t need to actually interpolate in the tables, but do compute the values you would need in order to enter the tables, and say which tables you must employ.)
2.18. A discrete variable $X$ can take only values 1, 2, 3, 4 and 5. Nevertheless, managers decide to “monitor process spread” using the ranges of samples of size $n = 2$. Suppose, for sake of argument, that under standard plant conditions observations are iid and uniform on the values 1 through 5 (i.e. $P[X = 1] = P[X = 2] = P[X = 3] = P[X = 4] = P[X = 5] = .2$).

(a) Find the distribution of $R$ for this situation. (Note that $R$ has possible values 0, 1, 2, 3 and 4. You need to reason out the corresponding probabilities.)

(b) The correct answer to part (a) has $E_R = 1.6$. This implies that if many samples of size $n = 2$ are taken and $R$ computed, one can expect a mean range near 1.6. Find and criticize corresponding normal theory control limits for $R$.

(c) Suppose that instead of using a normal-based Shewhart chart for $R$, one decides to use a high side Shewhart-CUSUM scheme (for ranges) with reference value $k_1 = 2$ and starting value 0, that signals the first time any range is 4 or the CUSUM is 3 or more. Use your answer for (a) and show how to find the ARL for this scheme. (You need not actually carry through the calculations, but show explicitly how to set things up.)

2.19. SQC novices faced with the task of analyzing a sequence of (say) $m$ individual observations collected over time often do the following: Compute “$\bar{x}$” and “$s$” from the $m$ data values and apply “control limits” $\bar{x} \pm 3s$ to the $m$ individuals. Say why this method of operation is essentially useless. (Compare Problem 1.8.)

2.20. Consider an $\bar{x}$ chart based on standards $\mu_0$ and $\sigma_0$ and samples of size $n$, where only the “one point outside 3$\sigma$ limits” alarm rule is in use.

(a) Find ARLs if in fact $\sigma = \sigma_0$, but $\sqrt{n} |\mu - \mu_0|/\sigma$ is respectively 0, 1, 2, and 3.

(b) Find ARLs if in fact $\mu = \mu_0$, but $\sigma/\sigma_0$ is respectively .5, .8, 1, 1.5 and 2.0.

Theory

2.21. Consider the problem of samples of size $n = 1$ in variables control charting contexts, and the notion of there using moving ranges for various purposes.
2. PROCESS MONITORING

This problem considers a little theory that may help illustrate the implications of using an average moving range, $\overline{MR}$, in the estimation of $\sigma$ in such circumstances.

Suppose that $X_1$ and $X_2$ are independent normal random variables with a common variance $\sigma^2$, but possibly different means $\mu_1$ and $\mu_2$. (You may, if you wish, think of these as widget diameters made at times 1 and 2, where the process mean has potentially shifted between the sampling periods.)

(a) What is the distribution of $X_1 - X_2$? The distribution of $(X_1 - X_2)/\sigma$?

(b) For $t > 0$, write out in terms of values of $\Phi$ the probability

$$P[(X_1 - X_2)/\sigma \leq t].$$

In doing this, abbreviate $(\mu_1 - \mu_2)/\sigma$ as $\delta$.

(c) Notice that in part (b), you have found the cumulative distribution function for the random variable $MR/\sigma$. Differentiate your answer to (b) to find the probability density for $MR/\sigma$ and then use this probability density to write down an integral that gives the mean of the random variable $MR/\sigma$, $E(MR/\sigma)$. (You may abbreviate the standard normal pdf as $\phi$, rather than writing everything out.)

Vardeman used his trusty HP 15C (and its definite integral routine) and evaluated the integral in (c) for various values of $\delta$. Some values that he obtained are below.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\pm 0$</th>
<th>$\pm 0.1$</th>
<th>$\pm 0.2$</th>
<th>$\pm 0.3$</th>
<th>$\pm 0.4$</th>
<th>$\pm 0.5$</th>
<th>$\pm 1.0$</th>
<th>$\pm 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(MR/\sigma)$</td>
<td>1.1284</td>
<td>1.1312</td>
<td>1.1396</td>
<td>1.1537</td>
<td>1.1732</td>
<td>1.198</td>
<td>1.399</td>
<td>1.710</td>
</tr>
<tr>
<td>$\pm 2.0$</td>
<td>$\pm 2.5$</td>
<td>$\pm 3.0$</td>
<td>$\pm 3.5$</td>
<td>$\pm 4.0$</td>
<td>large $</td>
<td>\delta</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>2.101</td>
<td>2.544</td>
<td>3.017</td>
<td>3.506</td>
<td>4.002</td>
<td>$</td>
<td>\delta</td>
<td>$</td>
<td></td>
</tr>
</tbody>
</table>

(Notice that as expected, the $\delta = 0$ value is $d_2$ for a sample of size $n = 2$.)

(d) Based on the information above, argue that for $n$ independent normal random variables $X_1, X_2, \ldots, X_n$ with common standard deviation $\sigma$, if $\mu_1 = \mu_2 = \cdots = \mu_n$ then the sample average moving range, $\overline{MR}$, when divided by 1.1284 has expected value $\sigma$.

(e) Now suppose that instead of being constant, the successive means, $\mu_1, \mu_2, \ldots, \mu_n$ in fact exhibit a reasonably strong linear trend. That is suppose that $\mu_t = \mu_{t-1} + \sigma$. What is the expected value of $\overline{MR}/1.1284$ in this situation. Does $\overline{MR}/1.1284$ seem like a sensible estimate of $\sigma$ here?
(f) In a scenario where the means could potentially “bounce around” according to $\mu_t = \mu_{t-1} \pm k\sigma$, how large might $k$ be without destroying the usefulness of $\overline{MR} / 1.1284$ as an estimate of $\sigma$? Defend your opinion on the basis of the information contained in the table above.

2.22. Consider the kind of discrete time Markov Chain with a single absorbing state used in §2.1 to study the run length properties of process monitoring schemes. Suppose that one wants to know not the mean times to absorption from the nonabsorbing states, but the variances of those times. Since for a generic random variable $X$, $\text{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$, once one has mean times to absorption (belonging to the vector $L = (I - R)^{-1}1$) it suffices to compute the expected squares of times to absorption. Let $M$ be an $m \times 1$ vector containing expected squares of times to absorption (from states $S_1$ through $S_m$). Set up a system of $m$ equations for the elements of $M$ in terms of the elements of $R$, $L$ and $M$. Then show that in matrix notation

$$M = (I - R)^{-1}(I + 2R(I - R)^{-1})1.$$ 

2.23. So-called “Stop-light Control” or “Target Area Control” of a measured characteristic $X$ proceeds as follows. One first defines “Green” (OK), “Yellow” (Marginal) and “Red” (Unacceptable) regions of possible values of $X$. One then periodically samples a process according to the following rules. At a given sampling period, a single item is measured and if it produces a Green $X$, no further action is necessary at the time period in question. If it produces a Red $X$, lack of control is declared. If it produces a Yellow $X$, a second item is immediately sampled and measured. If this second item produces a Green $X$, no further action is taken at the period in question, but otherwise lack of control is declared.

Suppose that in fact a process under stop-light monitoring is stable and $p_G = P[X \text{ is Green}]$, $p_Y = P[X \text{ is Yellow}]$ and $p_R = 1 - p_G - p_Y = P[X \text{ is Red}]$.

(a) Find the mean number of sampling periods from the beginning of monitoring through the first out-of-control signal, in terms of the $p$‘s.

(b) Find the mean total number of items measured from the beginning of monitoring through the first out-of-control signal, in terms of the $p$‘s.
2. PROCESS MONITORING

2.24. Consider the Run-Sum control chart scheme discussed in §2.2. In the notes Vardeman wrote out a transition matrix for a Markov Chain analysis of the behavior of this scheme.

(a) Write out the corresponding system of 8 linear equations in 8 mean times to absorption for the scheme. Note that the mean times till signal from “$T = -0$” and “$T = +0$” states are the same linear combinations of the 8 mean times and must thus be equal.

(b) Find a formula for the ARL of this scheme. This can be done as follows. Use the equations for the mean times to absorption from states “$T = +3$” and “$T = +2$” to find a constant $\kappa_{+2,+3}$ such that $L_{+3} = \kappa_{+2,+3} L_{+2}$. Find similar constants $\kappa_{+1,+2}$, $\kappa_{+0,+1}$, $\kappa_{-2,-3}$, $\kappa_{-1,-2}$ and $\kappa_{0,-1}$. Then use these constants to write a single linear equation for $L_{+0} = L_{-0}$ that you can solve for $L_{+0} = L_{-0}$.

2.25. Consider the problem of monitoring $X$ the number of nonconformities on a widget.

Suppose the standard for $\lambda$ is so small that a usual 3$\sigma$ Shewhart control chart will signal any time $X_t > 0$. On intuitive grounds the engineers involved find such a state of affairs unacceptable. The replacement for the standard Shewhart scheme that is then being contemplated is one that signals at time $t$ if

i) $X_t \geq 2$

or

ii) $X_t = 1$ and any of $X_{t-1}$, $X_{t-2}$, $X_{t-3}$ or $X_{t-4}$ is also equal to 1.

Show how you could find an ARL for this scheme. (Give either a matrix equation or system of linear equations one would need to solve. State clearly which of the quantities in your set-up is the desired ARL.)

2.26. Consider a discrete distribution on the (positive and negative) integers specified by the probability function $p(\cdot)$. This distribution will be used below to help predict the performance of a Shewhart type monitoring scheme that will sound an alarm the first time that an individual observation $X_t$ is 3 or more in absolute value (that is, the alarm bell rings the first time that $|X_t| \geq 3$).

(a) Give an expression for the ARL of the scheme in terms of values of $p(\cdot)$, if observations $X_1, X_2, X_3, \ldots$ are iid with probability function $p(\cdot)$. 

(b) Carefully set up and show how you would use a transition matrix for an appropriate Markov Chain in order to find the ARL of the scheme under a model for the observations $X_1, X_2, X_3, \ldots$ specified as follows:

$X_1$ has probability function $p(\cdot)$, and given $X_1, X_2, \ldots, X_{t-1}$, the variable $X_t$ has probability function $p(\cdot - X_{t-1})$

You need not carry out any matrix manipulations, but be sure to fully explain how you would use the matrix you set up.

2.27. Consider the problem of finding ARLs for a Shewhart individuals chart supposing that observations $X_1, X_2, X_3, \ldots$ are not iid, but rather realizations from a so-called AR(1) model. That is, suppose that in fact for some $\rho$ with $|\rho| < 1$

$$X_t = \rho X_{t-1} + \epsilon_t$$

for a sequence of iid normal random variables $\epsilon_1, \epsilon_2, \ldots$ each with mean 0 and variance $\sigma^2$. Notice that under this model the conditional distribution of $X_{t+1}$ given all previous observations is normal with mean $\rho X_t$ and variance $\sigma^2$.

Consider plotting values $X_t$ on a Shewhart chart with control limits $UCL$ and $LCL$.

(a) For $LCL < u < UCL$, let $L(u)$ stand for the mean number of additional observations (beyond $X_1$) that will be required to produce an out of control signal on the chart, given that $X_1 = u$. Carefully derive an integral equation for $L(u)$.

(b) Suppose that you can solve your equation from (a) for the function $L(u)$ and that it is sensible to assume that $X_1$ is normal with mean 0 and variance $\sigma^2/(1 - \rho^2)$. Show how you would compute the ARL for the Shewhart individuals chart under this model for the $X$ sequence.

2.28. A one-sided upper CUSUM scheme with reference value $k_1 = .5$ and decision interval $h_1 = 4$ is to be used to monitor Poisson ($\lambda$) observations. (CUSUM$\geq 4$ causes a signal.)

(a) Set up, but don’t try to manipulate with a Markov Chain transition matrix that you could use to find (exact) ARLs for this scheme.

(b) Set up, but don’t try to manipulate with a Markov Chain transition matrix that you could use to obtain (exact) ARLs if the CUSUM
scheme is combined with a Shewhart-type scheme that signals any time an observation 3 or larger is obtained.

2.29. In §2.3, Vardeman argued that if \( Q_1, Q_2, \ldots \) are iid continuous random variables with probability density \( f \) and cdf \( F \), a one-sided (high side) CUSUM scheme with reference value \( k_1 \) and decision interval \( h_1 \) has ARL function \( L(u) \) satisfying the integral equation

\[
L(u) = 1 + L(0) F(k_1 - u) + \int_0^{h_1} L(y) f(y + k_1 - u) dy.
\]

Suppose that a (one-sided) Shewhart type criterion is added to the CUSUM alarm criterion. That is, consider a monitoring system that signals the first time the high side CUSUM exceeds \( h_1 \) or \( Q_t > M \), for a constant \( M > k_1 \). Carefully derive an integral equation similar to the one above that must be satisfied by the ARL function of the combined Shewhart-CUSUM scheme.

2.30. Consider the problem of finding ARLs for CUSUM schemes where \( Q_1, Q_2, \ldots \) are iid exponential with mean 1. That is, suppose that one is CUSUMing iid random variables with common probability density

\[
f(x) = \begin{cases} 
  e^{-x} & \text{for } x > 0 \\
  0 & \text{otherwise}.
\end{cases}
\]

(a) Argue that the ARL function of a high side CUSUM scheme for this situation satisfies the differential equation

\[
L'(u) = \begin{cases} 
  L(u) - L(0) - 1 & \text{for } 0 \leq u \leq k_1 \\
  L(u) - L(u - k_1) - 1 & \text{for } k_1 \leq u.
\end{cases}
\]

(Vardeman and Ray (Technometrics, 1985) solve this differential equation and a similar one for low side CUSUMs to obtain ARLs for exponential \( Q \).)

(b) Suppose that one decides to approximate high side exponential CUSUM ARLs by using simple numerical methods to solve (approximately) the integral equation discussed in class. For the case of \( k_1 = 1.5 \) and \( h_1 = 4.0 \), write out the \( R \) matrix (in the equation \( L = 1 + RL \)) one has using the quadrature rule defined by \( m = 8 \), \( a_i = (2i - 1)h_1/2m \) and each \( w_i = h_1/m \).

(c) Consider making a Markov Chain approximation to the ARL referred to in part (b). For \( m = 8 \) and the discretization discussed in class, write out the \( R \) matrix that would be used in this case. How does this matrix compare to the one in part (b)?
2.31. Consider the problem of determining the run length properties of a high side CUSUM scheme with head start $u$, reference value $k$ and decision interval $h$ if iid continuous observations $Q_1, Q_2, \ldots$ with common probability density $f$ and cdf $F$ are involved. Let $T$ be the run length variable. In class, Vardeman concentrated on $L(u) = ET$, the ARL of the scheme. But other features of the run length distribution might well be of interest in some applications.

(a) The variance of $T$, $\text{Var } T = ET^2 - L^2(u)$ might also be of importance in some instances. Let $M(u) = ET^2$ and argue very carefully that $M(u)$ must satisfy the integral equation

$$M(u) = 1 + (M(0) + 2L(0)) F(k-u) + \int_0^h (M(s) + 2L(s)) f(s+k-u) ds .$$

(Once one has found $L(u)$, this gives an integral equation that can be solved for $M(u)$, leading to values for $\text{Var } T$, since then $\text{Var } T = M(u) - L^2(u)$.)

(b) The probability function of $T$, $P(t, u) = \Pr[T = t]$ might also be of importance in some instances. Express $P(1, u)$ in terms of $F$. Then argue very carefully that for $t > 1$, $P(t, u)$ must satisfy the recursion

$$P(t, u) = P(t - 1, 0) F(k-u) + \int_0^h P(t - 1, s) f(s + k - u) ds .$$

(There is thus the possibility of determining successively the function $P(1, u)$, then the function $P(2, u)$, then the function $P(3, u)$, etc.)

2.32. In §2.2, Vardeman considered a “two alarm rule monitoring scheme” due to Wetherill and showed how find the ARL for that scheme by solving two linear equations for quantities $L_1$ and $L_2$. It is possible to extend the arguments presented there and find the variance of the run length.

(a) For a generic random variable $X$, express both $\text{Var } X$ and $\text{E}(X + 1)^2$ in terms of $\text{E}X$ and $\text{E}X^2$.

(b) Let $M_1$ be the expected square of the run length for the Wetherill scheme and let $M_2$ be the expected square of the number of additional plotted points required to produce an out-of-control signal if there has been no signal to date and the current plotted point is between 2- and 3-sigma limits. Set up two equations for $M_1$ and $M_2$ that are linear in $M_1$, $M_2$, $L_1$ and $L_2$. 
2. PROCESS MONITORING

(c) The equations from (b) can be solved simultaneously for $M_1$ and $M_2$. Express the variance of the run length for the Wetherill scheme in terms of $M_1$, $M_2$, $L_1$ and $L_2$.

2.33. Consider a Shewhart control chart with the single extra alarm rule “signal if 2 out of any 3 consecutive points fall between 2σ and 3σ limits on one side of the center line.” Suppose that points $Q_1, Q_2, Q_3, \ldots$ are to be plotted on this chart and that the $Q$s are iid.

Use the notation

\begin{align*}
p_A &= \text{the probability } Q_1 \text{ falls outside } 3\sigma \text{ limits} \\
p_B &= \text{the probability } Q_1 \text{ falls between } 2\sigma \text{ and } 3\sigma \text{ limits above the center line} \\
p_C &= \text{the probability } Q_1 \text{ falls between } 2\sigma \text{ and } 3\sigma \text{ limits below the center line} \\
p_D &= \text{the probability } Q_1 \text{ falls inside } 2\sigma \text{ limits}
\end{align*}

and set up a Markov Chain that you can use to find the ARL of this scheme under the iid model for the $Q$s. (Be sure to carefully and completely define your state space, write out the proper transition matrix and indicate which entry of $(I - R)^{-1}1$ gives the desired ARL.)

2.34. A process has a “good” state and a “bad” state. Suppose that when in the good state, the probability that an observation on the process plots outside of control limits is $g$, while the corresponding probability for the bad state is $b$. Assume further that if the process is in the good state at time $t - 1$, there is a probability $d$ of degradation to the bad state before an observation at time $t$ is made. (Once the process moves into the bad state it stays there until that condition is detected via process monitoring and corrected.) Find the “ARL”/mean time of alarm, if the process is in the good state at time $t = 0$ and observation starts at time $t = 1$.

2.35. Consider the following (nonstandard) process monitoring scheme for a variable $X$ that has ideal value 0. Suppose $h(x) > 0$ is a function with $h(x) = h(-x)$ that is decreasing in $|x|$. ($h$ has its maximum at 0 and decreases symmetrically as one moves away from 0.) Then suppose that

i) control limits for $X_1$ are $\pm h(0)$,

and

ii) for $t > 1$ control limits for $X_t$ are $\pm h(X_{t-1})$.

(Control limits vary. The larger that $|X_{t-1}|$ is, the tighter are the limits on $X_t$.) Discuss how you would find an ARL for this scheme for iid $X$ with marginal probability density $f$. (Write down an appropriate integral
equation, briefly discuss how you would go about solving it and what you
would do with the solution in order to find the desired ARL.)

2.36. Consider the problem of monitoring integer-valued variables $Q_1, Q_2, Q_3, \ldots$ (we’ll suppose that $Q$ can take any integer value, positive or negative). Define

$$h(x) = 4 - |x|$$

and consider the following definition of an alarm scheme:

1) alarm at time $i = 1$ if $|Q_1| \geq 4$, and

2) for $i \geq 2$ alarm at time $i$ if $|Q_i| \geq h(Q_{i-1})$.

For integer $j$, let $q_j = P[Q_1 = j]$ and suppose the $Q_i$ are iid. Carefully describe how to find the ARL for this situation. (You don’t need to produce a formula, but you do need to set up an appropriate MC and tell me exactly/completely what to do with it in order to get the ARL.)

2.37. Consider the problem of monitoring integer-valued variables $Q_t$ (we’ll suppose that $Q$ can take any integer value, positive or negative). A combination of individuals and moving range charts will be used according to the scheme that at time 1, $Q_1$ alone will be plotted, while at time $t > 1$ both $Q_t$ and $MR_t = |Q_t - Q_{t-1}|$ will be plotted. The alarm will ring at the first period where $|Q_t| > 3$ or $MR_t > 4$. Suppose that the variables $Q_1, Q_2, \ldots$ are iid and $p_i = P[Q_1 = i]$. Consider the problem of finding an average run length in this scenario.

(a) Set up the transition matrix for an 8 state Markov Chain describing the evolution of this charting method from $t = 2$ onward, assuming that the alarm doesn’t ring at $t = 1$. (State $S_i$ for $i = -3, -2, -1, 0, 1, 2, 3$ will represent the situation “no alarm yet and the most recent observation is $i$” and there will be an alarm state.)

(b) Given values for the $p_i$, one could use the transition matrix from part (a) and solve for mean times to alarm from the states $S_i$. Call these $L_{-3}, L_{-2}, L_{-1}, L_0, L_1, L_2, \text{ and } L_3$. Express the average run length of the whole scheme (including the plotting at time $t = 1$ when only $Q_1$ is plotted) in terms of the $L_i$ and $p_i$ values.
3 Engineering Control and Stochastic Control Theory

3.1. Consider the use of the PI(D) controller $\Delta X(t) = .5E(t) + .25\Delta E(t)$ in a situation where the control gain, $G$, is 1 and the target for the controlled variable is $T(t) = 0$. Suppose that no control actions are applied before the time $t = 0$, but that for $t \geq 0$, $E(t)$ and $\Delta E(t)$ are used to make changes in the manipulated variable, $\Delta X(t)$, according to the above equation. Suppose further that the value of the controlled variable, $Y(t)$, is the sum of what the process would do with no control, say $Z(t)$, and the sum of effects at time $t$ of all changes in the manipulated variable made in previous periods based on $E(0)$, $\Delta E(0)$, $E(1)$, $\Delta E(1)$, $E(2)$, $\Delta E(2)$, ..., $E(t-1)$, $\Delta E(t-1)$.

Consider 3 possible patterns of impact at time $s$ of a change in the manipulated variable made at time $t$, $\Delta X(t)$:

Pattern 1: The effect on $Y(s)$ is $1 \times \Delta X(t)$ for all $s \geq t + 1$ (a control action takes its full effect immediately).

Pattern 2: The effect on $Y(t+1)$ is 0, but the effect on $Y(s)$ is $1 \times \Delta X(t)$ for all $s \geq t + 2$ (there is one period of dead time, after which a control action immediately takes its full effect).

Pattern 3: The effect on $Y(s)$ is $1 \times (1 - 2^{t-s})\Delta X(t)$ for all $s \geq t + 1$ (there is an exponential/geometric pattern in the way the impact of $\Delta X(t)$ is felt, the full effect only being seen for large $s$).

Consider also 3 possible deterministic patterns of uncontrolled process behavior, $Z(t)$:

Pattern A: $Z(t) = -3$ for all $t \geq -1$ (the uncontrolled process would remain constant, but off target).

Pattern B: $Z(t) = -3$ for all $-1 \leq t \leq 5$, while $Z(t) = 3$ for all $6 \leq t$ (there is a step change in where the uncontrolled process would be).

Pattern C: $Z(t) = -3 + t$ for all $t \geq -1$ (there is a linear trend in where the uncontrolled process would be).

For each of the $3 \times 3 = 9$ combinations of patterns in the impact of changes in the manipulated variable and behavior of the uncontrolled process, make up a table giving at times $t = -1, 0, 1, 2, \ldots, 10$ the values of $Z(t)$, $E(t)$, $\Delta E(t)$, $\Delta X(t)$ and $Y(t)$. 
3.2. Consider again the PI(D) controller of Problem 3.1. Suppose that the target is \( T(t) \), where \( T(t) = 0 \) for \( t \leq 5 \) and \( T(t) = 3 \) for \( t > 5 \). For the Pattern 1 of impact of control actions and Patterns A, B and C for \( Z(t) \), make up tables giving at times \( t = -1, 0, 1, 2, \ldots, 10 \) the values of \( Z(t) \), \( T(t) \), \( E(t) \), \( \Delta E(t) \), \( \Delta X(t) \) and \( Y(t) \).

3.3. Consider again the PI(D) controller of Problem 3.1 and Pattern D: \( Z(t) = (-1)^t \) (the uncontrolled process would oscillate around the target).

For the Patterns 1 and 2 of impact of control actions, make up tables giving at times \( t = -1, 0, 1, 2, \ldots, 10 \) the values of \( Z(t) \), \( T(t) \), \( E(t) \), \( \Delta E(t) \), \( \Delta X(t) \) and \( Y(t) \).

3.4. There are two tables here giving some values of an uncontrolled process \( Z(t) \) that has target \( T(t) = 0 \). Suppose that a manipulated variable \( X \) is available and that the simple (integral only) control algorithm

\[ \Delta X(t) = E(t) \]

will be employed, based on an observed process \( Y(t) \) that is the sum of \( Z(t) \) and the effects of all relevant changes in \( X \).

Consider two different scenarios:

(a) a change of \( \Delta X \) in the manipulated variable impacts all subsequent values of \( Y(t) \) by the addition of an amount \( \Delta X \), and

(b) there is one period of dead time, after which a change of \( \Delta X \) in the manipulated variable impacts all subsequent values of \( Y(t) \) by the addition of an amount \( \Delta X \).

Fill in the two tables according to these two scenarios and then comment on the lesson they seem to suggest about the impact of dead time on the effectiveness of PID control.

3.5. On pages 87 and 88 V&J suggest that over-adjustment of a process will increase rather than decrease variation. In this problem we will investigate this notion mathematically. Imagine periodically sampling a widget produced by a machine and making a measurement \( y_i \). Conceptualize the situation as

\[ y_i = \mu_i + \epsilon_i \]

where
### Table 6.1: Table for Problem 3.4(a), No Dead Time

<table>
<thead>
<tr>
<th>$t$</th>
<th>$Z(t)$</th>
<th>$T(t)$</th>
<th>$Y(t)$</th>
<th>$E(t) = \Delta X(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
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</tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-1</td>
<td>0</td>
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<td></td>
</tr>
</tbody>
</table>

### Table 6.2: Table for Problem 3.4(a), One Period of Dead Time

<table>
<thead>
<tr>
<th>$t$</th>
<th>$Z(t)$</th>
<th>$T(t)$</th>
<th>$Y(t)$</th>
<th>$E(t) = \Delta X(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
<tr>
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</tr>
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<tr>
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<td></td>
</tr>
</tbody>
</table>
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CHAPTER 6. PROBLEMS

\[ \mu_i = \text{the true machine setting (or widget diameter) at time } i \]
and
\[ \epsilon_i = \text{“random” variability at time } i \text{ affecting only measurement } i. \]

Further, suppose that the (coded) ideal diameter is 0 and \( \mu_i \) is the sum of natural machine drift and adjustments applied by an operator up through time \( i \). That is, with

\[ \gamma_i = \text{the machine drift between time } i - 1 \text{ and time } i \]
and
\[ \delta_i = \text{the operator (or automatic controller’s) adjustment applied between time } i - 1 \text{ and time } i \]
suppose that \( \mu_0 = 0 \) and for \( j \geq 1 \) we have

\[ \mu_j = \sum_{i=1}^{j} \gamma_i + \sum_{i=1}^{j} \delta_i. \]

We will here consider the (integral-only) adjustment policies for the machine

\[ \delta_i = -\alpha y_{i-1} \text{ for an } \alpha \in [0, 1]. \]

It is possible to verify that for \( j \geq 1 \)

- if \( \alpha = 0 \):
  \[ y_j = \sum_{i=1}^{j} \gamma_i + \epsilon_j \]
- if \( \alpha = 1 \):
  \[ y_j = \gamma_j - \epsilon_{j-1} + \epsilon_j \]
and
- if \( \alpha \in (0, 1) \):
  \[ y_j = \sum_{i=1}^{j} \gamma_i(1 - \alpha)^{j-i} - \alpha \sum_{i=1}^{j} \epsilon_{i-1}(1 - \alpha)^{j-i} + \epsilon_j. \]

Model \( \epsilon_0, \epsilon_1, \epsilon_2, \ldots \) as independent random variables with mean 0 and variance \( \sigma^2 \) and consider predicting the likely effectiveness of the adjustment policies by finding \( \lim_{j \to \infty} E\mu_j^2 \). (\( E\mu_j^2 \) is a measure of how close to proper adjustment the machine can be expected to be at time \( j \).)

(a) Compare choices of \( \alpha \) supposing that \( \gamma_i \geq 0 \). (Here the process is stable.)

(b) Compare choices of \( \alpha \) supposing that \( \gamma_i \geq d \), some constant. (This is a case of deterministic linear machine drift, and might for example be used to model tool wear over reasonably short periods.)

(c) Compare choices of \( \alpha \) supposing \( \gamma_1, \gamma_2, \ldots \) is a sequence of independent random variables with mean 0 and variance \( \eta^2 \) that is independent of the \( \epsilon \) sequence. What \( \alpha \) would you recommend using if this (random walk) model seems appropriate and \( \eta \) is thought to be about one half of \( \sigma \)?
3. ENGINEERING CONTROL AND STOCHASTIC CONTROL THEORY

3.6. Suppose that $\ldots, \epsilon(-1), \epsilon(0), \epsilon(1), \epsilon(2), \ldots$ are iid normal random variables with mean 0 and variance $\sigma^2$ and that

$$Z(t) = \epsilon(t-1) + \epsilon(t).$$

(Note that under this model consecutive $Z$’s are correlated, but those separated in time by at least 2 periods are independent.) As it turns out, under this model

$$E[F[Z(t+1)|Z(t)] = \frac{1}{t+2} \sum_{j=0}^{t} (-1)^j (t+1-j)Z(t-j)$$

while

$$E[F[Z(s)|Z(t)] = 0 \text{ for } s \geq t + 2.$$

If $T(t) \neq 0$ find optimal (MV) control strategies for two different situations involving numerical process adjustments $a$.

(a) First suppose that $A(a, s) = a$ for all $s \geq 1$. (Note that in the limit as $t \to \infty$, the MV controller is a “proportional-only” controller.)

(b) Then suppose the impact of a control action is similar to that in (a), except there is one period of delay, i.e.

$$A(a, s) = \begin{cases} 
  a & \text{for } s \geq 2 \\
  0 & \text{for } s = 1
\end{cases}$$

(You should decide that $a(t) = 0$ is optimal.)

(c) For the situation without dead time in part (a), write out $Y(t)$ in terms of $\epsilon$’s. What are the mean and variance of $Y(t)$? How do these compare to the mean and variance of $Z(t)$? Would you say from this comparison that the control algorithm is effective in directing the process to the target $T(t) = 0$?

(d) Again for the situation of part (a), consider the matter of process monitoring for a change from the model of this problem (that ought to be greeted by a revision of the control algorithm or some other appropriate intervention). Argue that after some start-up period it makes sense to Shewhart chart the $Y(t)’$s, treating them as essentially iid Normal $(0, \sigma^2)$ if “all is OK.” (What is the correlation between $Y(t)$ and $Y(t-1)$?)
3.7. Consider the optimal stochastic control problem as described in §3.1 with
\( Z(t) \) an iid normal \((0, 1)\) sequence of random variables, control actions
\( a \in (-\infty, \infty), \ A(a, s) = a \) for all \( s \geq 1 \) and \( T(s) = 0 \) for all \( s \). What do
you expect the optimal (minimum variance) control strategy to turn out
to be? Why?

3.8. (Vander Wiel) Consider a stochastic control problem with the following
elements. The (stochastic) model, \( \mathcal{F} \), for the uncontrolled process, \( Z(t) \),
will be
\[
Z(t) = \phi Z(t - 1) + \epsilon(t)
\]
where the \( \epsilon(t) \) are iid normal \((0, \sigma^2)\) random variables and \( \phi \) is a (known)
constant with absolute value less than 1. (\( Z(t) \) is a first order autoregress-
ive process.) For this model,
\[
E_{\mathcal{F}}[Z(t + 1)| \ldots, Z(-1), Z(0), Z(1), \ldots, Z(t)] = \phi Z(t) .
\]
For the function \( A(a, s) \) describing the effect of a control action \( a \) taken \( s \)
periods previous, we will use
\[
A(a, s) = a \rho^{s-1}
\]
for another known constant
\( 0 < \rho < 1 \) (the effect of an adjustment made at a given period dies out
geo\text{metrically}).

Carefully find \( a(0), a(1), \) and \( a(2) \) in terms of a constant target value \( T \)
and \( Z(0), Y(1) \) and \( Y(2) \). Then argue that in general
\[
a(t) = T \left( 1 + (\phi - \rho) \sum_{s=0}^{t-1} \phi^s \right) - \phi Y(t) - (\phi - \rho) \sum_{s=1}^{t} \phi^s Y(t - s) .
\]
For large \( t \), this prescription reduces to approximately what?

3.9. Consider the following stochastic control problem. The stochastic model,
\( \mathcal{F} \), for the uncontrolled process \( Z(t) \), will be
\[
Z(t) = ct + \epsilon(t)
\]
where \( c \) is a known constant and the \( \epsilon(t) \)'s are iid normal \((0, \sigma^2)\) random
variables. (The \( Z(t) \) process is a deterministic linear trend seen through iid/white noise.) For the function \( A(a, s) \) describing the effect of a control action \( a \) taken \( s \) periods previous, we will use
\[
A(a, s) = (1 - 2^{-s}) a
\]
for all \( s \geq 1 \). Suppose further that the target value for the controlled process is
\( T = 0 \) and that control begins at time 0 (after observing \( Z(0) \)).

(a) Argue carefully that
\[
\tilde{Z}(t) = E_{\mathcal{F}}[Z(t+1)| \ldots, Z(-1), Z(0), Z(1), \ldots, Z(t)] = c(t + 1).
\]
(b) Find the minimum variance control algorithm and justify your answer. Does there seem to be a limiting form for $a(t)$?

(c) According to the model here, the controlled process $Y(t)$ should have what kind of behavior? (How would you describe the joint distribution of the variables $Y(1), Y(2), \ldots, Y(t)$?) Suppose that you decide to set up Shewhart type “control limits” to use in monitoring the $Y(t)$ sequence. What values do you recommend for $LCL$ and $UCL$ in this situation? (These could be used as an on-line check on the continuing validity of the assumptions that we have made here about $F$ and $A(a,s)$.)

3.10. Consider the following optimal stochastic control problem. Suppose that for some (known) appropriate constants $\alpha$ and $\beta$, the uncontrolled process $Z(t)$ has the form

$$Z(t) = \alpha Z(t - 1) + \beta Z(t - 2) + \epsilon(t)$$

for the $\epsilon$’s iid with mean 0 and variance $\sigma^2$. (The $\epsilon$’s are independent of all previous $Z$’s.) Suppose further that for control actions $a \in (-\infty, \infty)$, $A(a, 1) = 0$ and $A(a, s) = a$ for all $s \geq 2$. (There is a one period delay, following which the full effect of a control action is immediately felt.) For $s \geq 1$, let $T(s)$ be an arbitrary sequence of target values for the process.

(a) Argue that

$$E_F[Z(t + 1)| \ldots, Z(t - 2), Z(t - 1), Z(t)] = \alpha Z(t) + \beta Z(t - 1)$$

and that

$$E_F[Z(t + 2)| \ldots, Z(t - 2), Z(t - 1), Z(t)] = (\alpha^2 + \beta)Z(t) + \alpha \beta Z(t - 1).$$

(b) Carefully find $a(0)$, $a(1)$ and $a(2)$ in terms of $Z(-1)$, $Z(0)$, $Y(1)$, $Y(2)$ and the $T(s)$ sequence.

(c) Finally, give a general form for the optimal control action to be taken at time $t \geq 3$ in terms of $\ldots, Z(-1), Z(0), Y(1), Y(2), \ldots, Y(t)$ and $a(0), a(1), \ldots, a(t - 1)$.

3.11. Use the first order autoregressive model of Problem 3.8 and consider the two functions $A(a, s)$ from Problem 3.6. Find the MV optimal control polices (in terms of the $Y$’s) for the $T \equiv 0$ situation. Are either of these PID control algorithms?
3.12. A process has a Good state and a Bad state. Every morning a gremlin tosses a coin with $P[\text{Heads}] = u > .5$ that governs how states evolve day to day. Let

$$C_i = P[\text{change state on day } i \text{ from that on day } i-1].$$

Each $C_i$ is either $u$ or $1 - u$.

(a) Before the gremlin tosses the coin on day $i$, you get to choose whether

$$C_i = u \text{ (so that Heads} \implies \text{change)}$$

or

$$C_i = 1 - u \text{ (so that Heads} \implies \text{no change)}$$

(You either apply some counter-measures or let the process evolve naturally.) Your object is to see that the process is in the Good state as often as possible. What is your optimal strategy? (What should you do on any morning $i$? This needs to depend upon the state of the process from day $i - 1$.)

(b) If all is as described here, the evolution of the states under your optimal strategy from (a) is easily described in probabilistic terms. Do so. Then describe in rough/qualitative terms how you might monitor the sequence of states to detect the possibility that the gremlin has somehow changed the rules of process evolution on you.

(c) Now suppose that there is a one-day time delay in your counter-measures. Before the gremlin tosses his coin on day you get to choose only whether

$$C_{i+1} = u$$

or

$$C_{i+1} = 1 - u.$$

(You do not get to choose $C_i$ on the morning of day $i$.) Now what is your optimal strategy? (What you should choose on the morning of day $i$ depends upon what you already chose on the morning of day $(i - 1)$ and whether the process was in the Good state or in the Bad state on day $(i - 1)$.) Show appropriate calculations to support your answer.
4. PROCESS CHARACTERIZATION

4 Process Characterization

4.1. The following are depth measurements taken on \( n = 8 \) pump end caps. The units are inches.

\[ 4.9991, 4.9990, 4.9994, 4.9986, 4.9991, 4.9993, 4.9990 \]

The specifications for this depth measurement were 4.999 ± 0.001 inches.

(a) As a means of checking whether a normal distribution assumption is plausible for these depth measurements, make a normal plot of these data. (Use regular graph paper and the method of Section 5.1.) Read an estimate of \( \sigma \) from this plot.

Regardless of the appearance of your plot from (a), henceforth suppose that one is willing to say that the process producing these lengths is stable and that a normal distribution of depths is plausible.

(b) Give a point estimate and a 90% two-sided confidence interval for the “process capability,” \( 6\sigma \).

(c) Give a point estimate and a 90% two-sided confidence interval for the process capability ratio \( C_p \).

(d) Give a point estimate and a 95% lower confidence bound for the process capability ratio \( C_{pk} \).

(e) Give a 95% two-sided prediction interval for the next depth measurement on a cap produced by this process.

(f) Give a 99% two-sided tolerance interval for 95% of all depth measurements of end caps produced by this process.

4.2. Below are the logarithms of the amounts (in ppm by weight) of aluminum found in 26 bihourly samples of recovered PET plastic at a Rutgers University recycling plant taken from a JQT paper by Susan Albin. (In this context, aluminum is an impurity.)

\[ 5.67, 5.40, 4.83, 4.37, 4.98, 4.78, 5.50, 4.77, 5.20, 4.14, 3.40, 4.94, 4.62, 4.62, 4.47, 5.21, 4.09, 5.25, 4.78, 6.24, 4.79, 5.15, 4.25, 3.40, 4.50, 4.74 \]

(a) Set up and plot charts for a sensible monitoring scheme for these values. (They are in order if one reads left to right, top to bottom.)
Caution: Simply computing a mean and sample standard deviation for these values and using “limits” for individuals of the form $\bar{x} \pm 3s$ does not produce a sensible scheme! Say clearly what you are doing and why.

(b) Suppose that (on the basis of an analysis of the type in (a) or otherwise) it is plausible to treat the 26 values above as a sample of size $n = 26$ from some physically stable normally distributed process. (Note $\bar{x} \approx 4.773$ and $s \approx .632$.)

i. Give a two-sided interval that you are “90% sure” will contain the next log aluminum content of a sample taken at this plant. Transform this to an interval for the next raw aluminum content.

ii. Give a two-sided interval that you are “95% sure” will contain 90% of all log aluminum contents. Transform this interval to one for raw aluminum contents.

(c) Rather than adopting the “stable process” model alluded to in part (b) suppose that it is only plausible to assume that the log purity process is stable for periods of about 10 hours, but that mean purities can change (randomly) at roughly ten hour intervals. Note that if one considers the first 25 values above to be 5 samples of size 5, some summary statistics are then given below:

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<th>period</th>
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<th>3</th>
<th>4</th>
<th>5</th>
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<td>$\bar{x}$</td>
<td>5.050</td>
<td>4.878</td>
<td>4.410</td>
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<td>.514</td>
<td>.590</td>
<td>.784</td>
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<tr>
<td>$R$</td>
<td>1.30</td>
<td>1.36</td>
<td>1.54</td>
<td>2.15</td>
<td>1.75</td>
</tr>
</tbody>
</table>

Based on the usual random effects model for this two-level “nested/hierarchical” situation, give reasonable point estimates of the within-period standard deviation and the standard deviation governing period to period changes in process mean.

4.3. A standard (in engineering statistics) approximation due to Wallis (used on page 468 of V&J) says that often it is adequate to treat the variable $\bar{x} \pm ks$ as if it were normal with mean $\mu \pm k\sigma$ and variance

$$\sigma^2 \left( \frac{1}{n} + \frac{k^2}{2n} \right).$$

Use the Wallis approximation to the distribution of $\bar{x} + ks$ and find $k$ such that for $x_1, x_2, \ldots, x_{26}$ iid normal random variables, $\bar{x} + ks$ is a 99% upper statistical tolerance bound for 95% of the population. (That is,
your job is to choose $k$ so that $P[\Phi\left(\frac{\bar{x}+ks-\mu}{\sigma}\right) \geq .95] \approx .99$.) How does your approximate value compare to the exact one given in Table A.9b?

4.4. Consider the problem of pooling together samples of size $n$ from, say, five different days to make inferences about all widgets produced during that period. In particular, consider the problem of estimating the fraction of widgets with diameters that are outside of engineering specifications. Suppose that

$$N_i = \text{the number of widgets produced on day } i$$

$$p_i = \text{the fraction of widgets produced on day } i \text{ that have diameters that are outside engineering specifications}$$

and

$$\hat{p}_i = \text{the fraction of the } i\text{th sample that have out-of-spec. diameters.}$$

If the samples are simple random samples of the respective daily productions, standard finite population sampling theory says that

$$E\hat{p}_i = p_i \quad \text{and} \quad \text{Var} \hat{p}_i = \left(\frac{N_i - 1}{N_i - n}\right) \frac{p_i(1-p_i)}{n}.$$ 

Two possibly different estimators of the population fraction of diameters out of engineering specifications,

$$p = \frac{\sum_{i=1}^{5} N_i p_i}{\sum_{i=1}^{5} N_i},$$

are

$$\hat{p} = \frac{\sum_{i=1}^{5} N_i \hat{p}_i}{\sum_{i=1}^{5} N_i} \quad \text{and} \quad \bar{p} = \frac{1}{5} \sum_{i=1}^{5} \hat{p}_i.$$ 

Show that $E\hat{p} = p$, but that $E\bar{p}$ need not be $p$ unless all $N_i$ are the same. Assuming the independence of the $\hat{p}_i$, what are the variances of $\hat{p}$ and $\bar{p}$? Note that neither of these needs to equal

$$\left(\frac{N - 1}{N - 5n}\right) \frac{p(1-p)}{5n}.$$
4.5. Suppose that the hierarchical random effects model used in Section 5.5 of V&J is a good description of how 500 widget diameters arise on each of 5 days in each of 10 weeks. (That is, suppose that the model is applicable with \( I = 10, J = 5 \) and \( K = 500 \).) Suppose further, that of interest is the grand (sample) variance of all \( 10 \times 5 \times 500 \) widget diameters. Use the expected mean squares and write out an expression for the expected value of this variance in terms of \( \sigma_\alpha^2, \sigma_\beta^2 \) and \( \sigma^2 \).

Now suppose that one only observes 2 widget diameters each day for 5 weeks and in fact obtains the “data” in the accompanying table. From these data obtain point estimates of the variance components \( \sigma_\alpha^2, \sigma_\beta^2 \) and \( \sigma^2 \). Use these and your formula from above to predict the variance of all \( 10 \times 5 \times 500 \) widget diameters. Then make a similar prediction for the variance of the diameters from the next 10 weeks, supposing that the \( \sigma_\alpha^2 \) variance component could be eliminated.

4.6. Consider a situation in which a lot of 50,000 widgets has been packed into 100 crates, each of which contains 500 widgets. Suppose that unbeknownst to us, the lot consists of 25,000 widgets with diameter 5 and 25,000 widgets with diameter 7. We wish to estimate the variance of the widget diameters in the lot (which is \( 50,000/49,999 \)). To do so, we decide to select 4 crates at random, and from each of those, select 5 widgets to measure.

(a) One (not so smart) way to try and estimate the population variance is to simply compute the sample variance of the 20 widget diameters we end up with. Find the expected value of this estimator under two different scenarios: 1st where each of the 100 crates contains 250 widgets of diameter 5 and 250 widgets with diameter 7, and then 2nd where each crate contains widgets of only one diameter. What, in general terms, does this suggest about when the naive sample variance will produce decent estimates of the population variance?

(b) Give the formula for an estimator of the population variance that is unbiased (i.e. has expected value equal to the population variance).

4.7. Consider the data of Table 5.8 in V&J and the use of the hierarchical normal random effects model to describe their generation.

(a) Find point estimates of the parameters \( \sigma_\alpha^2 \) and \( \sigma^2 \) based first on ranges and then on ANOVA mean squares.
### 4. PROCESS CHARACTERIZATION

Table 6.3: Data for Problem 4.5

<table>
<thead>
<tr>
<th>Day</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$\bar{y}_{ij}$</th>
<th>$s^2_{ij}$</th>
<th>$\bar{y}_i$</th>
<th>$s^2_{B_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>15.5</td>
<td>14.9</td>
<td>15.2</td>
<td>.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>15.2</td>
<td>15.2</td>
<td>15.2</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 1</td>
<td>W 14.2</td>
<td>14.2</td>
<td>14.2</td>
<td>0</td>
<td>15.0</td>
<td>.605</td>
</tr>
<tr>
<td>R</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>15.8</td>
<td>16.4</td>
<td>16.1</td>
<td>.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>6.2</td>
<td>7.0</td>
<td>6.6</td>
<td>.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>7.2</td>
<td>8.4</td>
<td>7.8</td>
<td>.72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 2</td>
<td>W 6.6</td>
<td>7.8</td>
<td>7.2</td>
<td>.72</td>
<td>7.0</td>
<td>.275</td>
</tr>
<tr>
<td>R</td>
<td>6.2</td>
<td>7.6</td>
<td>6.9</td>
<td>.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>5.6</td>
<td>7.4</td>
<td>6.5</td>
<td>1.62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>15.4</td>
<td>14.4</td>
<td>14.9</td>
<td>.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>13.9</td>
<td>13.3</td>
<td>13.6</td>
<td>.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 3</td>
<td>W 13.4</td>
<td>14.8</td>
<td>14.1</td>
<td>.98</td>
<td>14.0</td>
<td>.370</td>
</tr>
<tr>
<td>R</td>
<td>12.5</td>
<td>14.1</td>
<td>13.3</td>
<td>1.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>13.2</td>
<td>15.0</td>
<td>14.1</td>
<td>1.62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>10.9</td>
<td>11.3</td>
<td>11.1</td>
<td>.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>12.5</td>
<td>12.7</td>
<td>12.6</td>
<td>.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 4</td>
<td>W 12.3</td>
<td>11.7</td>
<td>12.0</td>
<td>.18</td>
<td>12.0</td>
<td>.515</td>
</tr>
<tr>
<td>R</td>
<td>11.0</td>
<td>12.0</td>
<td>11.5</td>
<td>.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>12.3</td>
<td>13.3</td>
<td>12.8</td>
<td>.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>7.5</td>
<td>6.7</td>
<td>7.1</td>
<td>.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>6.7</td>
<td>7.3</td>
<td>7.0</td>
<td>.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 5</td>
<td>W 7.2</td>
<td>6.0</td>
<td>6.6</td>
<td>.72</td>
<td>7.0</td>
<td>.155</td>
</tr>
<tr>
<td>R</td>
<td>7.6</td>
<td>7.6</td>
<td>7.6</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>6.3</td>
<td>7.1</td>
<td>6.7</td>
<td>.32</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(b) Find a standard error for your ANOVA-based estimator of \( \sigma_\alpha^2 \) from (a).

(c) Use the material in §1.5 and make a 90% two sided confidence interval for \( \sigma_\alpha^2 \).

4.8. All of the variance component estimation material presented in the text is based on balanced data assumptions. As it turns out, it is quite possible to do point estimation (based on sample variances) from even unbalanced data. A basic fact that enables this is the following: If \( X_1, X_2, \ldots, X_n \) are uncorrelated random variables, each with the same mean, then

\[
E s^2 = \frac{1}{n} \sum_{i=1}^{n} \text{Var} X_i.
\]

(Note that the usual fact that for iid \( X_i \), \( E s^2 = \sigma^2 \), is a special case of this basic fact.)

Consider the (hierarchical) random effects model used in Section 5.5 of the text. In notation similar to that in Section 5.5 (but not assuming that data are balanced), let

\[
\bar{y}_{ij} = \text{the sample mean of data values at level } i \text{ of } A \text{ and level } j \text{ of } B \text{ within } A
\]

\[
s_{ij}^2 = \text{the sample variance of the data values at level } i \text{ of } A \text{ and level } j \text{ of } B \text{ within } A
\]

\[
\bar{y}_i = \text{the sample mean of the values } \bar{y}_{ij} \text{ at level } i \text{ of } A
\]

\[
s_{B_i}^2 = \text{the sample variance of the values } \bar{y}_{ij} \text{ at level } i \text{ of } A
\]

and

\[
s_A^2 = \text{the sample variance of the values } \bar{y}_i
\]

Suppose that instead of being furnished with balanced data, one has a data set where 1) there are \( I = 2 \) levels of \( A \), 2) level 1 of \( A \) has \( J_1 = 2 \) levels of \( B \) while level 2 of \( A \) has \( J_2 = 3 \) levels of \( B \), and 3) level 1 of \( B \) within level 1 of \( A \) has \( n_{11} = 2 \) levels of \( C \), level 2 of \( B \) within level 1 of \( A \) has \( n_{12} = 4 \) levels of \( C \), levels 1 and 2 of \( B \) within level 2 of \( A \) have \( n_{21} = n_{22} = 2 \) levels of \( C \) and level 3 of \( B \) within level 2 of \( A \) has \( n_{23} = 3 \) levels of \( C \).

Evaluate the following: \( E s_{\text{pooled}}^2 \), \( E \left( \frac{1}{I} \sum_{i,j} s_{ij}^2 \right) \), \( E s_{B_1}^2 \), \( E s_{B_2}^2 \), \( E \left( s_{B_1}^2 + s_{B_2}^2 \right) \), \( E s_A^2 \). Then find linear combinations of \( s_{\text{pooled}}^2 \), \( \frac{1}{2} \left( s_{B_1}^2 + s_{B_2}^2 \right) \) and \( s_A^2 \) that could sensibly used to estimate \( \sigma_\beta^2 \) and \( \sigma_\alpha^2 \).
4.9. Suppose that on \( I = 2 \) different days (A), \( J = 4 \) different heats (B) of cast iron are studied, with \( K = 3 \) tests (C) being made on each. Suppose further that the resulting percent carbon measurements produce \( SSA = .0355, SSB(A) = .0081 \) and \( SSC(B(A)) = SSE = .4088 \).

(a) If one completely ignores the hierarchical structure of the data set, what “sample variance” is produced? Does this quantity estimate the variance that would be produced if on many different days a single heat was selected and a single test made? Explain carefully! (Find the expected value of the grand sample variance under the hierarchical random effects model and compare it to this variance of single measurements made on a single day.)

(b) Give point estimates of the variance components \( \sigma_A^2, \sigma_B^2 \) and \( \sigma^2 \).

(c) Your estimate of \( \sigma_A^2 \) should involve a linear combination of mean squares. Give the variance of that linear combination in terms of the model parameters and \( I, J \) and \( K \). Use that expression and propose a sensible estimated standard deviation (a standard error) for this linear combination. (See §1.4 and Problem 1.9.)

4.10. Consider the “one variable/second order” version of the “propagation of error” ideas discussed in Section 5.4 of the text. That is, for a random variable \( X \) with mean \( \mu \) and standard deviation \( \sigma^2 \) and “nice” function \( g \), let \( Y = g(X) \) and consider approximating \( EY \) and \( \text{Var} \, Y \). A second order approximation of \( g \) made at the point \( x = \mu \) is

\[
g(x) \approx g(\mu) + g'(\mu)(x - \mu) + \frac{1}{2}g''(\mu)(x - \mu)^2.
\]

(Note that the approximating quadratic function has the same value, derivative and second derivative as \( g \) for the value \( x = \mu \).) Let \( \kappa_3 = \text{E}(X - \mu)^3 \) and \( \kappa_4 = \text{E}(X - \mu)^4 \). Based on the above preamble, carefully argue for the appropriateness of the following approximations:

\[
EY \approx g(\mu) + \frac{1}{2}g''(\mu)\sigma^2
\]

and

\[
\text{Var} \, Y \approx (g'(\mu))^2\sigma^2 + g'(\mu)g''(\mu)\kappa_3 + \frac{1}{4}(g''(\mu))^2(\kappa_4 - \sigma^4).
\]

4.11. (Vander Wiel) A certain RCL network involving 2 resistors, 2 capacitors and a single inductor has a dynamic response characterized by the
"transfer function"

\[
\frac{V_{\text{out}}}{V_{\text{in}}}(s) = \frac{s^2 + \zeta_1 \omega_1 s + \omega_1^2}{s^2 + \zeta_2 \omega_2 s + \omega_2^2},
\]

where

\[
\omega_1 = (C_2 L)^{-1/2},
\]

\[
\omega_2 = \left(\frac{C_1 + C_2}{LC_1 C_2}\right)^{1/2},
\]

\[
\zeta_1 = \frac{R_2}{2L \omega_1},
\]

and

\[
\zeta_2 = \frac{R_1 + R_2}{2L \omega_2}.
\]

\(R_1\) and \(R_2\) are the resistances involved in ohms, \(C_1\) and \(C_2\) are the capacitances in Farads, and \(L\) is the value of the inductance in Henries. Standard circuit theory says that \(\omega_1\) and \(\omega_2\) are the "natural frequencies" of this network,

\[
\frac{\omega_1^2}{\omega_2^2} = \frac{C_1}{C_1 + C_2}
\]

is the "DC gain," and \(\zeta_1\) and \(\zeta_2\) determine whether the zeros and poles are real or complex. Suppose that the circuit in question is to be massed produced using components with the following characteristics:

\[
\begin{align*}
EC_1 &= \frac{1}{399} F \\
ER_1 &= 38 \\
EC_2 &= \frac{1}{2} F \\
ER_2 &= 2 \\
EL &= 1 H
\end{align*}
\]

\[
\begin{align*}
\text{Var} C_1 &= \left(\frac{1}{399}\right)^2 \\
\text{Var} R_1 &= (3.8)^2 \\
\text{Var} C_2 &= \left(\frac{1}{20}\right)^2 \\
\text{Var} R_2 &= (.2)^2 \\
\text{Var} L &= (.1)^2
\end{align*}
\]

Treat \(C_1, R_2, C_2, R_2\) and \(L\) as independent random variables and use the propagation of error approximations to do the following:

(a) Approximate the mean and standard deviation of the DC gains of the manufactured circuits.

(b) Approximate the mean and standard deviation of the natural frequency \(\omega_2\).
Now suppose that you are designing such an RCL circuit. To simplify things, use the capacitors and the inductor described above. You may choose the resistors, but their quality will be such that

$$\text{Var } R_1 = (ER_1/10)^2 \quad \text{and} \quad \text{Var } R_2 = (ER_2/10)^2.$$ 

Your design goals are that $\zeta_2$ should be (approximately) .5, and subject to this constraint, $\text{Var } \zeta_2$ be minimum.

(c) What values of $ER_1$ and $ER_2$ satisfy (approximately) the design goals, and what is the resulting (approximate) standard deviation of $\zeta_2$?

(Hint for part (c): The first design goal allows one to write $ER_2$ as a function of $ER_1$. To satisfy the second design goal, use the propagation of error idea to write the (approximate) variance of $\zeta_2$ as a function of $ER_1$ only. By the way, the first design goal allows you to conclude that none of the partial derivatives needed in the propagation of error work depend on your choice of $ER_1$.)

4.12. Manufacturers wish to produce autos with attractive “fit and finish,” part of which consists of uniform (and small) gaps between adjacent pieces of sheet metal (like, e.g., doors and their corresponding frames). The accompanying figure is an idealized schematic of a situation of this kind, where we (at least temporarily) assume that edges of both a door and its frame are linear. (The coordinate system on this diagram is pictured as if its axes are “vertical” and “horizontal.” But the line on the body need not be an exactly “vertical” line, and whatever this line’s intended orientation relative to the ground, it is used to establish the coordinate system as indicated on the diagram.)

On the figure, we are concerned with gaps $g_1$ and $g_2$. The first is at the level of the top hinge of the door and the second is $d$ units “below” that level in the body coordinate system ($d$ units “down” the door frame line from the initial measurement). People manufacturing the car body are responsible for the dimension $w$. People stamping the doors are responsible for the angles $\theta_1$ and $\theta_2$ and the dimension $y$. People welding the top door hinge to the door are responsible for the dimension $x$. And people hanging the door on the car are responsible for the angle $\phi$. The quantities $x, y, w, \phi, \theta_1$ and $\theta_2$ are measurable and can be used in manufacturing to
verify that the various folks are “doing their jobs.” A door design engineer has to set nominal values for and produce tolerances for variation in these quantities. This problem is concerned with how the propagation of errors method might help in this tolerancing enterprise, through an analysis of how variation in \(x, y, w, \phi, \theta_1\) and \(\theta_2\) propagates to \(g_1, g_2\) and \(g_1 - g_2\).

If I have correctly done my geometry/trigonometry, the following relationships hold for labeled points on the diagram:

\[
p = (-x \sin \phi, x \cos \phi)
\]
\[
q = p + (y \cos \left(\phi + \left(\theta_1 - \frac{\pi}{2}\right)\right), y \sin \left(\phi + \left(\theta_1 - \frac{\pi}{2}\right)\right))
\]
\[
s = (q_1 + q_2 \tan (\phi + \theta_1 + \theta_2 - \pi), 0)
\]
and
\[
u = (q_1 + (q_2 + d) \tan (\phi + \theta_1 + \theta_2 - \pi), -d)
\]

Then for the idealized problem here (with perfectly linear edges) we have
\[
g_1 = w - s_1
\]
4. PROCESS CHARACTERIZATION

and

\[ g_2 = w - u_1 \, . \]

Actually, in an attempt to allow for the notion of “form error” in the ideally linear edges, one might propose that at a given distance “below” the origin of the body coordinate system the realized edge of a real geometry is its nominal position plus a “form error.” Then instead of dealing with \( g_1 \) and \( g_2 \), one might consider the gaps

\[ g_1^* = g_1 + \epsilon_1 - \epsilon_2 \]

and

\[ g_2^* = g_2 + \epsilon_3 - \epsilon_4 \, , \]

for body form errors \( \epsilon_1 \) and \( \epsilon_3 \) and door form errors \( \epsilon_2 \) and \( \epsilon_4 \). (The interpretation of additive “form errors” around the line of the body door frame is perhaps fairly clear, since “the error” at a given level is measured perpendicular to the “body line” and is thus well-defined for a given realized body geometry. The interpretation of an additive error on the right side “door line” is not so clear, since in general one will not be measuring perpendicular to the line of the door, or even at any consistent angle with it. So for a realized geometry, what “form error” to associate with a given point on the ideal line or exactly how to model it is not completely clear. We’ll ignore this logical problem and proceed using the models above.)

We’ll use \( d = 40 \) cm, and below are two possible sets of nominal values for the parameters of the door assembly:

<table>
<thead>
<tr>
<th>Design A</th>
<th>Design B</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 20 ) cm</td>
<td>( x = 20 ) cm</td>
</tr>
<tr>
<td>( y = 90 ) cm</td>
<td>( y = 90 ) cm</td>
</tr>
<tr>
<td>( w = 90.4 ) cm</td>
<td>( w = (90 \cos \frac{\pi}{10} + .4) ) cm</td>
</tr>
<tr>
<td>( \phi = 0 )</td>
<td>( \phi = \frac{\pi}{10} )</td>
</tr>
<tr>
<td>( \theta_1 = \frac{\pi}{2} )</td>
<td>( \theta_1 = \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( \theta_2 = \frac{\pi}{4} )</td>
<td>( \theta_2 = \frac{\pi}{10} )</td>
</tr>
</tbody>
</table>

Partial derivatives of \( g_1 \) and \( g_2 \) (evaluated at the design nominal values of \( x, y, w, \phi, \theta_1 \) and \( \theta_2 \)) are:
(a) Suppose that a door engineer must eventually produce tolerances for \( x, y, w, \phi, \theta_1 \) and \( \theta_2 \) that are consistent with “\( \pm 1 \) cm” tolerances on \( g_1 \) and \( g_2 \). If we interpret “\( \pm 1 \) cm” tolerances to mean \( \sigma_{g_1} \) and \( \sigma_{g_2} \) are no more than 0.033 cm, consider the set of “sigmas”

\[
\begin{align*}
\sigma_x &= .01 \text{ cm} \\
\sigma_y &= .01 \text{ cm} \\
\sigma_w &= .01 \text{ cm} \\
\sigma_\phi &= .001 \text{ rad} \\
\sigma_{\theta_1} &= .001 \text{ rad} \\
\sigma_{\theta_2} &= .001 \text{ rad}
\end{align*}
\]

First for Design A and then for Design B, investigate whether this set of “sigmas” is consistent with the necessary final tolerances on \( g_1 \) and \( g_2 \) in two different ways. Make propagation of error approximations to \( \sigma_{g_1} \) and \( \sigma_{g_2} \). Then simulate 100 values of both \( g_1 \) and \( g_2 \) using independent normal random variables \( x, y, w, \phi, \theta_1 \) and \( \theta_2 \) with means equal to the design nominals and these standard deviations. (Compute the sample standard deviations of the simulated values and compare to the .033 cm target.)

(b) One of the assumptions standing behind the propagation of error approximations is the independence of the input random variables. Briefly discuss why independence of the variables \( \theta_1 \) and \( \theta_2 \) may not be such a great model assumption in this problem.

(c) Notice that for Design A the propagation of error formula predicts that variation on the dimension \( x \) will not much affect the gaps presently of interest, \( g_1 \) and \( g_2 \), while the situation is different for
Design B. Argue, based on the nominal geometries, that this makes perfectly good sense. For Design A, one might say that the gaps $g_1$ and $g_2$ are "robust" to variation in $x$. For this design, do you think that the entire "fit" of the door to the body of the car is going to be "robust to variation in $x$"? Explain.

(Note, by the way, that the fact that $\frac{\partial g_1}{\partial \phi} = 0$ for Design A also makes this design look completely "robust to variation in $\phi$" in terms of the gap $g_1$, at least by standards of the propagation of error formula. But the situation for this variable is somewhat different than for $x$. This partial derivative is equal to 0 because for $y, w, \phi, \theta_1$ and $\theta_2$ at their nominal values, $g_1$ considered as a function of $\phi$ alone has a local minimum at $\phi = 0$. This is different from $g_1$ being constant in $\phi$. A more refined "second order" propagation of error analysis of this problem, that essentially begins from a quadratic approximation to $g_1$ instead of a linear one, would distinguish between these two possibilities. But the "first order" analysis done on the basis of formula (5.27) of the text is often helpful and adequate for practical purposes.)

(d) What does the propagation of error formula predict for variation in the difference $g_1 - g_2$, first for Design A, and then for Design B?

(e) Suppose that one desires to take into account the possibility of "form errors" affecting the gaps, and thus considers analysis of $g_1^\ast$ and $g_2^\ast$ instead of $g_1$ and $g_2$. If standard deviations for the variables $\epsilon$ are all .001 cm, what does the propagation of error analysis predict for variability in $g_1^\ast$ and $g_2^\ast$ for Design A?

4.13. The electrical resistivity, $\rho$, of a wire is a property of the material involved and the temperature at which it is measured. At a given temperature, if a cylindrical piece of wire of length $L$ and (constant) cross-sectional area $A$ has resistance $R$, then the material’s resistivity is calculated as

$$\rho = \frac{RA}{L}.$$

In a lab exercise intended to determine the resistivity of copper at 20$^\circ$C, students measure the length, diameter and resistance of a wire assumed to have circular cross-sections. Suppose the length is approximately 1 meter, the diameter is approximately $2.0 \times 10^{-3}$ meters and the resistance is approximately $0.54 \times 10^{-2}$. Suppose further that the precisions of the
measuring equipment used in the lab are such that standard deviations $\sigma_L = 10^{-3}$ meter, $\sigma_D = 10^{-4}$ meter and $\sigma_R = 10^{-4}$ are appropriate.

(a) Find an approximate standard deviation that might be used to describe the precision associated with an experimentally derived value of $\rho$.

(b) Imprecision in which of the measurements appears to be the biggest contributor to imprecision in experimentally determined values of $\rho$? (Explain.)

(c) One should probably expect the approximate standard deviation derived here to under-predict the kind of variation that would actually be observed in such lab exercises over a period of years. Explain why this is so.

4.14. A bullet is fired horizontally into a block (of much larger mass) suspended by a long cord, and the impact causes the block and embedded bullet to swing upward a distance $d$ measured vertically from the block’s lowest position. The laws of mechanics can be invoked to argue that if $d$ is measured in feet, and before testing the block weighs $w_1$, while the block and embedded bullet together weigh $w_2$ (in the same units), then the velocity (in fps) of the bullet just before impact with the block is approximately

$$v = \left( \frac{w_2}{w_2 - w_1} \right) \sqrt{64.4 \cdot d}.$$ 

Suppose that the bullet involved weighs about .05 lb, the block involved weighs about 10.00 lb and that both $w_1$ and $w_2$ can be determined with a standard deviation of about .005 lb. Suppose further that the distance $d$ is about .50 ft, and can be determined with a standard deviation of .03 ft.

(a) Compute an approximate standard deviation describing the uncertainty in an experimentally derived value of $v$.

(b) Would you say that the uncertainties in the weights contribute more to the uncertainty in $v$ than the uncertainty in the distance? Explain.

(c) Say why one should probably think of calculations like those in part (a) as only providing some kind of approximate lower bound on the uncertainty that should be associated with the bullet’s velocity.

4.15. On page 243 of V&J there is an ANOVA table for a balanced hierarchical data set. Use it in what follows.
5. SAMPLING INSPECTION

(a) Find standard errors for the usual ANOVA estimates of $\sigma^2_\alpha$ and $\sigma^2$ (the “casting” and “analysis” variance components).

(b) If you were to later make 100 castings, cut 4 specimens from each of these and make a single lab analysis on each specimen, give a (numerical) prediction of the overall sample variance of these future 400 measurements (based on the hierarchical random effects model and the ANOVA estimates of $\sigma^2_\alpha$, $\sigma^2_\beta$ and $\sigma^2$).

5 Sampling Inspection

Methods

5.1. Consider attributes single sampling.

(a) Make type A OC curves for $N = 20$, $n = 5$ and $c = 0$ and 1, for both percent defective and mean defects per unit situations.

(b) Make type B OC curves for $n = 5$, $c = 0$, 1 and 2 for both percent defective and mean defects per unit situations.

(c) Use the imperfect inspection analysis presented in §5.2 and find OC bands for the percent defective cases above with $c = 1$ under the assumption that $w_D \leq .1$ and $w_G \leq .1$.

5.2. Consider single sampling for percent defective.

(a) Make approximate OC curves for $n = 100$, $c = 1$; $n = 200$, $c = 2$; and $n = 300$, $c = 3$.

(b) Make AOQ and ATI curves for a rectifying inspection scheme using a plan with $n = 200$ and $c = 2$ for lots of size $N = 10,000$. What is the AOQL?

5.3. Find attributes single sampling plans (i.e. find $n$ and $c$) having approximately

(a) $Pa = .95$ if $p = .01$ and $Pa = .10$ if $p = .03$.

(b) $Pa = .95$ if $p = 10^{-6}$ and $Pa = .10$ if $p = 3 \times 10^{-6}$.

5.4. Consider a (truncated sequential) attributes acceptance sampling plan, that for $X_n =$ the number of defective items found through the $n$th item inspected
rejects the lot if it ever happens that $X_n \geq 1.5 + .5n$, accepts the lot if it ever happens that $X_n \leq -1.5 + .5n$, and further never samples more than 11 items. We will suppose that if sampling were extended to $n = 11$, we would accept for $X_{11} = 4$ or 5 and reject for $X_{11} = 6$ or 7 and thus note that sampling can be curtailed at $n = 10$ if $X_{10} = 4$ or 6.

(a) Find expressions for the OC and ASN for this plan.
(b) Find formulas for the AOQ and ATI of this plan, if it is used in a rectifying inspection scheme for lots of size $N = 100$.

5.5. Consider single sampling based on a normally distributed variable.

(a) Find a single limit variables sampling plan with $L = 1.000$, $\sigma = .015$, $p_1 = .03$, $P_{a_1} = .95$, $p_2 = .10$ and $P_{a_2} = .10$. Sketch the OC curve of this plan. How does $n$ compare with what would be required for an attributes sampling plan with a comparable OC curve?
(b) Find a double limits variables sampling plan with $L = .49$, $U = .51$, $\sigma = .004$, $p_1 = .03$, $P_{a_1} = .95$, $p_2 = .10$ and $P_{a_2} = .10$. Sketch the OC curve of this plan. How does $n$ compare with what would be required for an attributes sampling plan with a comparable OC curve?
(c) Use the Wallis approximation and find a single limit variables sampling plan for $L = 1.000$, $p_1 = .03$, $P_{a_1} = .95$, $p_2 = .10$ and $P_{a_2} = .10$. Sketch an approximate OC curve for this plan.

5.6. In contrast to what you found in Problem 5.3(b), make use of the fact that the upper $10^{-6}$ point of the standard normal distribution is about 4.753, while the upper $3 \times 10^{-6}$ point is about 4.526 and find the $n$ required for a known $\sigma$ single limit variables acceptance sampling plan to have $P_a = .95$ if $p = 10^{-6}$ and $P_a = .10$ if $p = 3 \times 10^{-6}$. What is the Achilles heel (fatal weakness) of these calculations?

5.7. Consider the CSP-1 plan with $i = 100$ and $f = .02$. Make AFI and AOQ plots for this plan and find the AOQL for both cases where defectives are rectified and where they are culled.

5.8. Consider the classical problem of acceptance sampling plan design. Suppose that one wants plans whose OC “drops” near $p = .03$ (wants $P_a \approx .5$ for $p = .03$) also wants $p = .04$ to have $P_a \approx .05$. 
5.10. Consider the situation of a consumer who will repeatedly receive lots of 1500 assemblies. These assemblies may be tested at a cost of $24 apiece or simply be put directly into a production stream with a later extra manufacturing cost of $780 occurring for each defective that is undetected because it was not tested. We'll assume that the supplier replaces any assembly found to be defective (either at the testing stage or later when the extra $780 cost occurs) with a guaranteed good assembly at no additional cost to the consumer. Suppose further that the producer of the assemblies has agreed to establish statistical control with $p = .02$.

(a) Adopt perspective B with $p$ known to be .02 and compare the mean per-lot costs of the following 3 policies:

i. test the whole lot,

ii. test none of the lot, and

iii. go to Mil. Std. 105D with AQL = .025 and adopt an inspection level II, normal inspection single sampling plan (i.e. $n = 125$ and $c = 7$), doing 100% inspection of rejected lots. (This by the
way, is not a recommended “use” of the standard. It is designed
to “guarantee” a consumer the desired AQL only when all the
switching rules are employed. I’m abusing the standard.)

(b) Adopt the point of view that in the short term, perspective B may
be appropriate, but that over the long term the supplier’s $p$ vacillates
between .02 and .04. In fact, suppose that for successive lots the

$$p_i = \text{perspective B} \quad p\text{ at the time lot } i \text{ is produced}$$

are independent random variables, with $P[p_i = .02] = P[p_i = .04] = .5$. Now compare the mean costs of policies i), ii) and iii) from (a)
used repeatedly.

(c) Suppose that the scenario in (b) is modified by the fact that the
consumer gets control charts from the supplier in time to determine
whether for a given lot, perspective B with $p = .02$ or $p = .04$ is
appropriate. What should the consumer’s inspection policy be, and
what is its mean cost of application?

5.11. Suppose that the fractions defective in successive large lots of fixed size $N$
can be modeled as iid Beta $(\alpha, \beta)$ random variables with $\alpha = 1$
and $\beta = 9$. Suppose that these lots are subjected to attributes acceptance sampling,
using $n = 100$ and $c = 1$. Find the conditional distribution of $p$ given that
the lot is accepted. Sketch probability densities for both the original Beta
distribution and this conditional distribution of $p$ given lot acceptance.

5.12. Consider the following variation on the “Deming Inspection Problem” dis-
cussed in §5.3. Each item in an incoming lot of size $N$ will be Good (G),
Marginal (M) or Defective (D). Some form of (single) sampling inspection
is contemplated based on counts of G’s, D’s and M’s. There will be a
per-item inspection cost of $k_1$ for any item inspected, while any M’s go-
ing uninspected will eventually produce a cost of $k_2$, and any D’s going
uninspected will produce a cost of $k_3 > k_2$. Adopt perspective B, i.e. that
any given incoming lot was produced under some set of stable conditions,
characterized here by probabilities $p_G$, $p_M$ and $p_D$ that any given item in
that lot is respectively G, M or D.

(a) Argue carefully that the “All or None” criterion is in force here and
identify the condition on the $p$’s under which “All” is optimal and
the condition under which “None” is optimal.
(b) If \( p_G, p_M \) and \( p_D \) are not known, but rather are described by a joint probability distribution, \( n \) other than \( N \) or 0 can turn out to be optimal. A particularly convenient distribution to use in describing the \( p \)'s is the Dirichlet distribution (it is the multivariate generalization of the Beta distribution for variables that must add up to 1). For a Dirichlet distribution with parameters \( \alpha_G > 0, \alpha_M > 0 \) and \( \alpha_D > 0 \), it turns out that if \( X_G, X_M \) and \( X_D \) are the counts of \( G \)'s, \( M \)'s and \( D \)'s in a sample of \( n \) items, then

\[
E[p_G|X_G, X_M, X_D] = \frac{\alpha_G + X_G}{\alpha_G + \alpha_M + \alpha_D + n} \\
E[p_M|X_G, X_M, X_D] = \frac{\alpha_M + X_M}{\alpha_G + \alpha_M + \alpha_D + n}
\]

and

\[
E[p_D|X_G, X_M, X_D] = \frac{\alpha_D + X_D}{\alpha_G + \alpha_M + \alpha_D + n}.
\]

Use these expressions and describe what an optimal lot disposal (acceptance or rejection) is, if a Dirichlet distribution is used to describe the \( p \)'s and a sample of \( n \) items yields counts \( X_G, X_M \) and \( X_D \).

5.13. Consider the Deming Inspection Problem exactly as discussed in §5.3. Suppose that \( k_1 = \$50, k_2 = \$500, N = 200 \) and one's a priori beliefs are such that one would describe \( p \) with a (Beta) distribution with mean .1 and standard deviation .090453. For what values of \( n \) are respectively \( c = 0, 1 \) and 2 optimal? If you are brave (and either have a pretty good calculator or are fairly quick with computing) compute the expected total costs associated with these values of \( n \) (obtained using the corresponding \( c^{opt}(n) \)). From these calculations, what \((n, c)\) pair appears to be optimal?

5.14. Consider the problem of estimating the process fraction defective based on the results of an “inverse sampling plan” that samples until 2 defective items have been found. Find the UMVUE of \( p \) in terms of the random variable \( n \) = the number of items required to find the second defective. Show directly that this estimator of \( p \) is unbiased (i.e. has expected value equal to \( p \)). Write out a series giving the variance of this estimator.

5.15. The paper “The Economics of Sampling Inspection” by Bernard Smith (that appeared in Industrial Quality Control in 1965 and is based on earlier theoretical work of Guthrie and Johns) gives a closed form expression for an approximately optimal \( n \) in the Deming inspection problem for cases
where \( p \) has a Beta(\( \alpha, \beta \)) prior distribution and both \( \alpha \) and \( \beta \) are integers. Smith says

\[
n^{\text{opt}} \approx \sqrt{\frac{N \cdot B(\alpha, \beta)p_0^\alpha (1-p_0)^\beta}{2 \left( p_0 \text{Bi}(\alpha|\alpha + \beta - 1, p_0) - \frac{\alpha}{\alpha + \beta} \text{Bi}(\alpha + 1|\alpha + \beta, p_0) \right)}}
\]

for \( p_0 \equiv k_1/k_2 \) the break-even quantity, \( B(\cdot, \cdot) \) the usual beta function and \( \text{Bi}(x|n, p) \) the probability that a binomial \((n, p)\) random variable takes a value of \( x \) or more. Suppose that \( k_1 = 50, k_2 = 500, N = 200 \) and our \textit{a priori} beliefs about \( p \) (or the “process curve”) are such that it is sensible to describe \( p \) as having mean .1 and standard deviation .090453. What fixed \( n \) inspection plan follows from the Smith formula?

5.16. Consider the Deming inspection scenario as discussed in §5.3. Suppose that \( N = 3, k_1 = 1.5, k_2 = 10 \) and a prior distribution \( G \) assigns \( P[p = .1] = .5 \) and \( P[p = .2] = .5 \). Find the optimal fixed \( n \) inspection plan by doing the following.

(a) For sample sizes \( n = 1 \) and \( n = 2 \), determine the corresponding optimal acceptance numbers, \( c^{\text{opt}}_n \).

(b) For sample sizes \( n = 0, 1, 2 \) and \( 3 \) find the expected total costs associated with those sample sizes if corresponding best acceptance numbers are used.

5.17. Consider the Deming inspection scenario once again. With \( N = 100, k_1 = 1 \) and \( k_2 = 10 \), write out the fixed \( p \) expected total cost associated with a particular choice of \( n \) and \( c \). Note that “None” is optimal for \( p < .1 \) and “All” is optimal for \( p > .1 \). So, in some sense, what is exactly optimal is highly discontinuous in \( p \). On the other hand, if \( p \) is “near” .1, it doesn’t matter much what inspection plan one adopts, “All,” “None” or anything else for that matter. To see this, write out as a function of \( p \)

\[
\frac{\text{worst possible expected total cost}(p) - \text{best possible expected total cost}(p)}{\text{best possible expected total cost}(p)}
\]

How big can this quantity get, e.g., on the interval \([.09, .11]\)?

5.18. Consider the following percent defective acceptance sampling scheme. One will sample items one at a time up to a maximum of 8 items. If at any point in the sampling, half or more of the items inspected are defective, sampling will cease and the lot will be rejected. If the maximum 8 items are inspected without rejecting the lot, the lot will be accepted.
5. SAMPLING INSPECTION

(a) Find expressions for the type B Operating Characteristic and the ASN of this plan.

(b) Find an expression for the type A Operating Characteristic of this plan if lots of \( N = 50 \) items are involved.

(c) Find expressions for the type B AOQ and ATI of this plan for lots of size \( N = 50 \).

(d) What is the (uniformly) minimum variance unbiased estimator of \( p \) for this plan? (Say what value one should estimate for every possible stop-sampling point.)

5.19. Vardeman argued in §5.3 that if one adopts perspective B with known \( p \) and costs are assessed as the sum of identically calculated costs associated with individual items, either “All” or “None” inspection plans will be optimal. Consider the following two scenarios (that lack one or the other of these assumptions) and show that in each the “All or None” paradigm fails to hold.

(a) Consider the Deming inspection scenario discussed in §5.3, with \( k_1 = \$1 \) and \( k_2 = \$100 \) and suppose lots of \( N = 5 \) are involved. Suppose that one adopts not perspective B, but instead perspective A, and that \( p \) is known to be .2 (a lot contains exactly 1 defective). Find the expected total costs associated with “All” and then with “None” inspection. Then suggest a sequential inspection plan that has smaller expected total cost than either “All” or “None.” (Find the expected total cost of your suggested plan and verify that it is smaller than that for both “All” and “None” inspection plans.)

(b) Consider perspective B with \( p \) known to be .4. Suppose lots of size \( N = 5 \) are involved and costs are assessed as follows. Each inspection costs $1 and defective items are replaced with good items at no charge. If the lot fails to contain at least one good item (and this goes undetected) a penalty of $1000 will be incurred, but otherwise the only costs charged are for inspection. Find the expected total costs associated with “All” and then with “None” inspection. Then argue convincingly that there is a better “fixed \( n \)” plan. (Say clearly what plan is superior and show that its expected total cost is less than both “All” and “None” inspection.)

5.20. Consider the following nonstandard “variables” acceptance sampling situation. A supplier has both a high quality/low variance production line
CHAPTER 6. PROBLEMS

(1) and a low quality/high variance production line (2) used to manufacture widgets ordered by Company V. Coded values of a critical dimension of these widgets produced on the high quality line are normally distributed with \( \mu_1 = 0 \) and \( \sigma_1 = 1 \), while coded values of this dimension produced on the low quality line are normally distributed with \( \mu_2 = 0 \) and \( \sigma_2 = 2 \). Coded specifications for this dimension are \( L = -3 \) and \( U = 3 \). The supplier is known to mix output from the two lines in lots sent to Company V. As a cost saving measure, this is acceptable to Company V, provided the fraction of “out-of-spec.” widgets does not become too large. Company V expects \( \pi \) the proportion of items in a lot coming from the high variance line (2) to vary lot to lot and decides to institute a kind of incoming variables acceptance sampling scheme. What will be done is the following. The critical dimension, \( X \), will be measured on each of \( n \) items sampled from a lot. For each measurement \( X \), the value \( Y = X^2 \) will be calculated. Then, for a properly chosen constant, \( k \), the lot will be accepted if \( Y \leq k \) and rejected if \( Y > k \). The purpose of this problem is to identify suitable \( n \) and \( k \), if \( Pa \approx .95 \) is desired for lots with \( p = .01 \) and \( Pa \approx .05 \) is desired for lots with \( p = .03 \).

(a) Find an expression for \( p \) (the long run fraction defective) as a function of \( \pi \). What values of \( \pi \) correspond to \( p = .01 \) and \( p = .03 \) respectively?

(b) It is possible to show (you need not do so here) that \( EY = 3\pi + 1 \) and \( \text{Var} Y = -9\pi^2 + 39\pi + 2 \). Use these facts, your answer to (a) and the Central Limit Theorem to help you identify suitable values of \( n \) and \( k \) to use at Company V.

5.21. On what basis is it sensible to criticize the relevance of the calculations usually employed to characterize the performance of continuous sampling plans?

5.22. Individual items produced on a manufacturer’s line may be graded as “Good” (G), “Marginal” (M) or “Defective” (D). Under stable process conditions, each successive item is (independently) G with probability \( p_G \), M with probability \( p_M \) and D with probability \( p_D \), where \( p_G + p_M + p_D = 1 \). Suppose that ultimately, defective items cause three times as much extra expense as marginal ones.

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5. SAMPLING INSPECTION

Based on the kind of cost information alluded to above, one might give each inspected item a “score” $s$ according to

$$s = \begin{cases} 
3 & \text{if the item is D} \\
1 & \text{if the item is M} \\
0 & \text{if the item is G} 
\end{cases}$$

It is possible to argue (don’t bother to do so here) that $E{s} = 3p_D + p_M$ and $\text{Var } s = 9p_D(1 - p_D) + p_M(1 - p_M) - 3p_Dp_M$.

(a) Give formulas for standards-given Shewhart control limits for average scores $\bar{s}$ based on samples of size $n$. Describe how you would obtain the information necessary to calculate limits for future control of $\bar{s}$.

(b) Ultimately, suppose that “standard” values are set at $p_G = .90$, $p_M = .07$ and $p_D = .03$ and $n = 100$ is used for samples of a high volume product. Use a normal approximation to the distribution of $\bar{s}$ and find an approximate ARL for your scheme from part (a) if in fact the mix of items shifts to where $p_G = .85$, $p_M = .10$ and $p_D = .05$.

(c) Suppose that one decides to use a high side CUSUM scheme to monitor individual scores as they come in one at a time. Consider a scheme with $k_1 = 1$ and no head-start that signals the first time that a CUSUM of scores of at least $h_1 = 6$ is reached. Set up an appropriate transition matrix and say how you would use that matrix to find an ARL for this scheme for an arbitrary set of probabilities $(p_G, p_M, p_D)$.

(d) Suppose that inspecting an item costs 1/5th of the extra expense caused by an undetected marginal item. A plausible (single sampling) acceptance sampling plan for lots of $N = 10,000$ of these items then accepts the lot if

$$\bar{s} \leq .20.$$ 

If rejection of the lot will result in 100% inspection of the remainder, consider the (“perspective B”) economic choice of sample size for plans of this form, in particular the comparison of $n = 100$ and $n = 400$ plans. The following table gives some approximate acceptance probabilities for these plans under two sets of probabilities $p = (p_G, p_M, p_D)$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$n = 100$</th>
<th>$n = 400$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(.9, .07, .03)$</td>
<td>$Pa \approx .76$</td>
<td>$Pa \approx .92$</td>
</tr>
<tr>
<td>$(.85, .10, .05)$</td>
<td>$Pa \approx .24$</td>
<td>$Pa \approx .08$</td>
</tr>
</tbody>
</table>
Find expected costs for these two plans \((n = 100\) and \(n = 400\)) if costs are accrued on a per-item and per-inspection basis and “prior” probabilities of these two sets of process conditions are respectively .8 for \(p = (.9,.07,.03)\) and .2 for \(p = (.85,.10,.05)\).

5.23. Consider variables acceptance sampling for a quantity \(X\) that has engineering specifications \(L = 3\) and \(U = 5\). We will further suppose that \(X\) has standard deviation \(\sigma = .2\).

(a) Suppose that \(X\) is uniformly distributed with mean \(\mu\). That is, suppose that \(X\) has probability density

\[
f(x) = \begin{cases} 1.4434 & \text{if } \mu - .3464 < x < \mu + .3464 \\ 0 & \text{otherwise} \end{cases}
\]

What means \(\mu_1\) and \(\mu_2\) correspond to fractions defective \(p_1 = .01\) and \(p_2 = .03\)?

(b) Find a sample size \(n\) and number \(k\) such that a variables acceptance sampling plan that accepts a lot when \(4 - k < \bar{x} < 4 + k\) and rejects it otherwise, has \(Pa_1 \approx .95\) for \(p_1 = .01\) and \(Pa_2 \approx .10\) for \(p_2 = .03\) when, as in part (a), observations are uniformly distributed with mean \(\mu\) and standard deviation \(\sigma = .2\).

(c) Suppose that one applies your plan from (b), but instead of being uniformly distributed with mean \(\mu\) and standard deviation \(\sigma = .2\), observations are normal with that mean and standard deviation. What acceptance probability then accompanies a fraction defective \(p_1 = .01\)?

5.24. A large lot of containers are each full of a solution of several gases. Suppose that in a given container the fraction of the solution that is gas A can be described with the probability density

\[
f(x) = \begin{cases} (\theta + 1)x^\theta & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}
\]

For this density, it is possible to show that \(E X = (\theta + 1)/(\theta + 2)\) and \(\text{Var} X = (\theta + 1)/(\theta + 2)^2(\theta + 3)\). Containers with \(X < .1\) are considered defective and we wish to do acceptance sampling to hopefully screen lots with large \(p\).

(a) Find the values of \(\theta\) corresponding to fractions defective \(p_1 = .01\) and \(p_2 = .03\).
(b) Use the Central Limit Theorem and find a number $k$ and a sample size $n$ so that an acceptance sampling plan that rejects if $\bar{x} < k$ has $P_{a_1} = .95$ and $P_{a_2} = .10$.

5.25. A measurement has an upper specification $U = 5.0$. Making a normal distribution assumption with $\sigma = .015$ and desiring $P_{a_1} = .95$ for $p_1 = .03$ and $P_{a_2} = .10$ for $p_2 = .10$, a statistician sets up a variables acceptance sampling plan for a sample of size $n = 23$ that rejects a lot if $\bar{x} > 4.97685$. In fact, a Weibull distribution with shape parameter $\beta = 400$ and scale parameter $\alpha$ is a better description of this characteristic than the normal distribution the statistician used. This alternative distribution has cdf

$$F(x|\alpha) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \exp\left(-\left(\frac{x}{\alpha}\right)^{400}\right) & \text{if } x > 0 , \end{cases}$$

and mean $\mu \approx .9986\alpha$ and standard deviation $\sigma = .0032\alpha$.

Show how to obtain an approximate OC curve for the statistician’s acceptance sampling plan under this Weibull model. (Use the Central Limit Theorem.) Use your method to find the real acceptance probability if $p = .03$.

5.26. Here’s a prescription for a possible fraction nonconforming attributes acceptance sampling plan:

stop and reject the lot the first time that $X_n \geq 2 + \frac{\hat{p}}{2}$

stop and accept the lot the first time that $n - X_n \geq 2 + \frac{\hat{p}}{2}$

(a) Find a formula for the OC for this “symmetric wedge-shaped plan.”

(One never samples more than 7 items and there are exactly 8 stop sampling points prescribed by the rules above.)

(b) Consider the use of this plan where lots of size $N = 100$ are subjected to rectifying inspection and inspection error is possible. (Assume that any item inspected and classified as defective is replaced with one drawn from a population that is in fact a fraction $p$ defective and has been inspected and classified as good.) Use the parameters $w_G$ and $w_D$ defined in §5.2 of the notes and give a formula for the real AOQ of this plan as a function of $p$, $w_G$ and $w_D$.

5.27. Consider a “perspective A” economic analysis of some fraction defective “fixed n inspection plans.” (Don’t simply try to use the type B calculations made in class. They aren’t relevant. Work this out from first principles.)
Suppose that \( N = 10 \), \( k_1 = 1 \) and \( k_2 = 10 \) in a “Deming Inspection Problem” cost structure. Suppose further that a “prior” distribution for \( p \) (the actual lot fraction defective) places equal probabilities on \( p = 0, .1 \) and \( .2 \). Here we will consider only plans with \( n = 0, 1 \) or \( 2 \). Let

\[ X = \text{the number of defectives in a simple random sample from the lot} \]

(a) For \( n = 1 \), find the conditional distributions of \( p \) given \( X = x \).

For \( n = 2 \), it turns out that the joint distribution of \( X \) and \( p \) is:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>.333</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.2</td>
<td>.267</td>
<td>.067</td>
<td>0</td>
</tr>
<tr>
<td>.333</td>
<td>.207</td>
<td>.119</td>
<td>.007</td>
</tr>
<tr>
<td>.333</td>
<td>.807</td>
<td>.185</td>
<td>.007</td>
</tr>
</tbody>
</table>

and the conditionals of \( p \) given \( X = x \) are:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>.413</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.2</td>
<td>.330</td>
<td>.360</td>
<td>0</td>
</tr>
<tr>
<td>.2</td>
<td>.2257</td>
<td>.640</td>
<td>1.00</td>
</tr>
</tbody>
</table>

(b) Use your answer to (a) and show that the best \( n = 1 \) plan REJECTS if \( X = 0 \) and ACCEPTS if \( X = 1 \). (Yes, this is correct!) Then use the conditionals above for \( n = 2 \) and show that the best \( n = 2 \) plan REJECTS if \( X = 0 \) and ACCEPTS if \( X = 1 \) or \( 2 \).

(c) Standard acceptance sampling plans REJECT FOR LARGE \( X \). Explain in qualitative terms why the best plans from (b) are not of this form.

(d) Which sample size \( (n = 0, 1 \) or \( 2) \) is best here? (Show calculations to support your answer.)