Graduate Lectures and Problems in Quality Control and Engineering Statistics: Theory and Methods

To Accompany

Statistical Quality Assurance Methods for Engineers

by

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A Useful Probabilistic Approximation

Here we present the general "delta method" or "propagation of error" approximation that stands behind several variance approximations in these notes as well as much of §5.4 of V&J. Suppose that a $p \times 1$ random vector

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$$

has a mean vector

$$\boldsymbol{\mu} = \begin{pmatrix} EX_1 \\ EX_2 \\ \vdots \\ EX_p \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

and $p \times p$ variance-covariance matrix

$$\begin{aligned} \boldsymbol{\Sigma} &= \begin{pmatrix} \mathrm{Var}X_1 & \mathrm{Cov}\,(X_1, X_2) & \cdots & \mathrm{Cov}\,(X_1, X_{p-1}) & \mathrm{Cov}\,(X_1, X_p) \\ \mathrm{Cov}\,(X_1, X_2) & \mathrm{Var}X_2 & \cdots & \mathrm{Cov}\,(X_2, X_{p-1}) & \mathrm{Cov}\,(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{Cov}\,(X_1, X_{p-1}) & \mathrm{Cov}\,(X_2, X_{p-1}) & \cdots & \mathrm{Var}X_{p-1} & \mathrm{Cov}\,(X_{p-1}, X_p) \\ \mathrm{Cov}\,(X_1, X_p) & \mathrm{Cov}\,(X_2, X_p) & \cdots & \mathrm{Cov}\,(X_{p-1}, X_p) & \mathrm{Var}X_p \end{pmatrix} \\ &= \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1,p-1}\sigma_1\sigma_{p-1} & \rho_{1p}\sigma_1\sigma_p \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2,p-1}\sigma_2\sigma_{p-1} & \rho_{2p}\sigma_2\sigma_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{2p}\sigma_2\sigma_p & \rho_{2,p-1}\sigma_2\sigma_{p-1} & \cdots & \sigma_{p-1}^2 & \rho_{p-1,p}\sigma_{p-1}\sigma_p \\ \rho_{1p}\sigma_1\sigma_p & \rho_{2p}\sigma_2\sigma_p & \cdots & \rho_{p-1,p}\sigma_{p-1}\sigma_p & \sigma_p^2 \end{pmatrix} \\ &= \left(\rho_{ij}\sigma_i\sigma_j\right) \end{aligned}$$

(Recall that if X_1 and X_j are independent, $\rho_{ij} = 0$.)

Then for a $k \times p$ matrix of constants

$$\boldsymbol{A} = (a_{ij})$$

consider the random vector

$$\underset{k\times 1}{\boldsymbol{Y}} = \underset{k\times p}{\boldsymbol{A}}\,\underset{p\times 1}{\boldsymbol{X}}$$

It is a standard piece of probability that $\boldsymbol{Y}$ has mean vector

$$\begin{pmatrix} EY_1 \\ EY_2 \\ \vdots \\ EY_k \end{pmatrix} = \boldsymbol{A}\,\boldsymbol{\mu}$$

and variance-covariance matrix

$$\text{Cov}\,\boldsymbol{Y} = \boldsymbol{A}\,\boldsymbol{\Sigma}\,\boldsymbol{A}'$$

(The $k = 1$ version of this for uncorrelated X_i is essentially quoted in (5.23) and (5.24) of V&J.)

The propagation of error method says that if instead of the relationship $\boldsymbol{Y} = \boldsymbol{A}\,\boldsymbol{X}$, I concern myself with k functions $g_1, g_2, ..., g_k$ (each mapping $\mathbf{R}^p$ to $\mathbf{R}$) and define

$$\boldsymbol{Y} = \begin{pmatrix} g_1(\boldsymbol{X}) \\ g_2(\boldsymbol{X}) \\ \vdots \\ g_k(\boldsymbol{X}) \end{pmatrix}$$

a multivariate Taylor's Theorem argument and the facts above provide an approximate mean vector and an approximate covariance matrix for $\boldsymbol{Y}$. That is, if the functions g_i are differentiable, let

$$\underset{k\times p}{\boldsymbol{D}} = \left(\left. \frac{\partial g_i}{\partial x_j} \right|_{\mu_1,\mu_2,\ldots,\mu_p} \right)$$

A multivariate Taylor approximation says that for each x_i near μ_i

$$\boldsymbol{y} = \begin{pmatrix} g_1(\boldsymbol{x}) \\ g_2(\boldsymbol{x}) \\ \vdots \\ g_k(\boldsymbol{x}) \end{pmatrix} \approx \begin{pmatrix} g_1(\boldsymbol{\mu}) \\ g_2(\boldsymbol{\mu}) \\ \vdots \\ g_k(\boldsymbol{\mu}) \end{pmatrix} + \boldsymbol{D}\,(\boldsymbol{x} - \boldsymbol{\mu})$$

So if the variances of the X_i are small (so that with high probability $\boldsymbol{Y}$ is near $\boldsymbol{\mu}$, that is that the linear approximation above is usually valid) it is plausible that $\boldsymbol{Y}$ has mean vector

$$\begin{pmatrix} EY_1 \\ EY_2 \\ \vdots \\ EY_k \end{pmatrix} \approx \begin{pmatrix} g_1(\boldsymbol{\mu}) \\ g_2(\boldsymbol{\mu}) \\ \vdots \\ g_k(\boldsymbol{\mu}) \end{pmatrix}$$

and variance-covariance matrix

$$\text{Cov}\,\boldsymbol{Y} \approx \boldsymbol{D}\,\boldsymbol{\Sigma}\,\boldsymbol{D}'$$