For a reference value $k_1$, some starting value $U_0 = u \geq 0$ (sometimes called the "headstart") and some "decision interval" $h_1 > 0$ let

$$U_i = \max \left[ 0, U_{i-1} + (Q_i - k_1) \right]$$

and signal the first time $U_i$ exceeds $h_1$ - this is a CUSUM with a "restart" feature (to be used if it falls negative.) That signals when it exceeds $h_1$.

$V_0 = v \leq 0$ and a decision interval $h_2 \geq 0$. Let

$$V_i = \min \left[ 0, V_{i-1} + (Q_i - k_2) \right]$$

and signal when $V_i$ first falls below $-h_2$ - this is meant to catch the eventuality of "small"/"decreased" $M_0$.

Combined schemes (simultaneous high- and low-side schemes) can be used to monitor for any change in $M_0$. 

Better: Consideration of High- and Low-Side Decision Interval CUSUM schemes.

High Side "Decision Interval" CUSUM Scheme

Low-Side "Decision Interval" CUSUM Scheme.

This is meant to catch the eventualty of "large"/"increased" $M_0$.
MC Methods for CUSUM ARLs

MC = Minimum Cost Utility Scheme

Tools:
1. i = sufficient statistic
2. MC's stopping times
3. Valued equations

Why is it true that MC's behavior of ARLs is usually suficient for a given set of parameters and model?

Fact (Yechin) \( R(\alpha/2,1,0) = 5 \).

Procedures:
1. Split the test into high and low self-decision intervals with starting values \( u \) and \( v \).
2. Use a MC with states:
   - M: \( S_i = \text{null} \)
   - H: \( S_i = \text{alarm} \)
3. For \( i = 0, 1, 2, 3, \ldots, m-1 \):
   - Update the state
   - \( S_i = \text{null} \)
   - \( S_i = \text{alarm} \)
4. When \( S_i = \text{alarm} \):
   - \( i = \text{endpoint} \)
   - \( i = \text{endpoint} \)

Answer is in the range of 4.2 at V=5.
if I can write down an approximate \( P \) then \( L_i \) is an approximate ARL using headstart \( i \left( \frac{A}{m} \right) \) — i.e.

\[(I-R)^{-1} 1 = L \rightarrow \text{entries are ARL's} \]

\( P? \) (see next page)

\[ L_0, L_1, L_2, \ldots, L_{m-1} \]

\[ L_i \approx \text{ARL from } i \left( \frac{A}{m} \right) \text{ headstart} \]

\[ L_i(u) = \text{ARL for a high-side scheme with headstart } u \text{ (and decision interval } h_i \text{ and reference value } k_i) \]

\[ L_i(u) = 1 \cdot P \left[ Q_i - k_i \geq h_i - u \right] + (1+L_i(0)) P \left[ Q_i - k_i \leq u \right] \]

\[ + \int_{k_i-h_i-u}^{u} (1+L_i(u+t-k_i))f(t)dt \]

Numerical Solutions of Integral Equations to Produce ARLs

Here we will again consider iid \( Q_i \) with marginal density \( f \)

\[ L_i(u) = 1 + L_i(0) F(k_i-u) + \int_0^{h_i} L_i(t)f(t+k_i-u)dt \]

? Solutions? Usually one must resort to numerical methods ... This involves approximating the integral.

Suppose I wish to approximate integrals on \([a, \infty] \) for "reasonable" functions \( g \) (i.e. I'm interested in \( \int_a^\infty g(t)dt \)) — I can choose a
quadrature method consisting of points
\[ a = a_1 \leq a_2 \leq a_3 \ldots \leq a_m \leq a + h \]
and weights
\[ w_i \geq 0 \quad \text{with} \quad \sum w_i = h \]
and approximate
\[ \int_a^b g(t) \, dt \approx \sum_{i=1}^m w_i g(a_i) \]

given an m point quadrature rule for \([0, h]\) I can write
\[ L_1(u) \approx 1 + \sum_{j=1}^m w_j L_1(a_j) f(a_j + k_1 - u) \]
If I knew \( L_1(a_1), L_1(a_2), \ldots, L_1(a_m) \)
I would have an approximate form for the function \( L_1(u) \)
\[ \forall \text{solve for them} \]
e.g.
\[ a_i = a + \frac{i \cdot h}{m} \]
and \[ w_i = \frac{h}{m} \]
gives the simple/crude "histogram" approximation to the integral

use this expression at \( u = a_1, a_2, \ldots, a_m \)
to get a set of linear equations
for \( L_1(a_1), L_1(a_2), \ldots, L_1(a_m) \)
\[ L_1(a_i) \approx 1 + \sum_{j=1}^m w_j L_1(a_j) f(a_j + k_1 - a_i) \]
m equations
it turns out that one can write this set of \( n \) equations in the form

\[
L = 1 + RL
\]

where now \( R \) is not quite the MC R matrix but is mighty close — see development on pages 31–33 of notes for \( L, R \). Note that entries in columns 2 through \( m \) of this \( R \) look like those in MC analysis. Why?

So the 2 approaches are nearly the same at the end of the day — one benefit derived from the presentation of the present analysis is a nice interpolation formula —

\[
L_i(u) \approx 1 + L_i(a) F(k_i - u) + \sum_{j=1}^{m} w_j L_j(a) f(a + k_i - u)
\]

Obviously the normal & version of this is important — note that the following 3 CUSUMs (high-side) are identical in behavior.

Suppose I'm using

\[
a_i = \frac{(i-s)}{m} h_i \quad w_i = \frac{h_i}{m}
\]

Then a generic entry (row \( i \) column \( j \)) here is

\[
w_j f^*(a_j - a_i + k_i) = \frac{h_i}{m} f^* \left( \frac{h_i}{m} (j-i) + k_i \right)
\]

\[
\approx \frac{h_i}{m} \left( \frac{h_i}{m} (j-i) \right) \frac{1}{2m} f^* \left( \frac{1}{2m} (j-i) \right)
\]

\[
\approx \frac{h_i}{m} \left( \frac{h_i}{m} (j-i) \right) \int f^*(t) dt = \frac{h_i}{m} \left( \frac{h_i}{m} (j-i) \right)
\]

<table>
<thead>
<tr>
<th>CUSUMmed</th>
<th>reference value</th>
<th>decision interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_i )</td>
<td>( k_l )</td>
<td>( h_1 )</td>
</tr>
<tr>
<td>( Q_i - Q_{i-1} )</td>
<td>( k_{1-k_1} )</td>
<td>( h_1 )</td>
</tr>
<tr>
<td>( Q_i - M_{Q} )</td>
<td>( K_{1-M_Q} )</td>
<td>( h_1 )</td>
</tr>
<tr>
<td>( Q_i - M_{Q} )</td>
<td>( K_{1-M_Q} )</td>
<td>( h_1 )</td>
</tr>
</tbody>
</table>
So "clearly"

\[ N(\mu_0, \sigma^2) \] properties of CUSUM at \( Q_i \) with reference value \( k_1 \) and decision interval \( h_1 \)

\[ \text{N(0,1) properties of a scheme with} \]

\[ \text{standardized reference} = \frac{k_1 - \mu_0}{\sigma} = -\Delta \]

(see 4.16)

\[ \text{decision interval} = \frac{h_1}{\sigma} = \Delta^* \]

Note that for \( N(0,1) \) case

1) Gan's program provides CUSUM ARL's (and quantiles of the run length distribution) for one-sided schemes

2) Table A.4 of V+J gives O-Hausdorf ARL's (from Gan)

In the notation of 4.15-4.17 of V+J

\[
ARL \approx \begin{cases} 
(\delta^* + 1.166)^2 & \text{if } \delta^* = 0 \\
\exp \left[ -2 \delta^* (\delta^* + 1.166) \right] - 1 + 2 \delta^* (\delta^* + 1.166) & \text{if } \delta^* \neq 0 \\
\frac{z(\delta^*)^2}{\delta^*} & \text{if } \delta^* \neq 0
\end{cases}
\]