Balanced 2-Way Factorial Data are typically used in Gage R&R contexts.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td></td>
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<td>4</td>
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</tbody>
</table>

Section 2.2.2 of V+J does range-based estimation of

\[ \tau \]

\[ \tau_{\text{repeatability}} = \sqrt{\tau_a^2 + \tau_{\text{repeatability}}^2} \]

\[ \tau_{\text{overall}} = \tau_{\text{R+R}} = \sqrt{\tau^2 + \tau_{\text{repeatability}}^2} \]

A beta methodology is to make use of 2-way ANOVA and the associated mean squares — necessary facts are summarized in § 1.4 of notes and V+V.

Under model (2.4) of V+J

MSA, MSB, MSAB, MSE are independent sample variance like objects, and quantities of interest are again of form

\[ \Theta = \varphi(\text{EMSA}, \text{EMSB}, \text{EMSAB}, \text{EMSE}) \]

and natural point estimators are

\[ U = \varphi(\text{NSA}, \text{NSB}, \text{NSAB}, \text{MSE}) \]

and Burdick + Graybill material can be used to make CI’s —

😊 Andy Chiang has implemented many useful parts of this (for Gage R&R studies)

An important point about this material is that you will quickly learn that standard corporate norms for Gage R&R studies prescribe very bad experimental designs — I=10, J=2 or 3, and m=3.
Read the balance of ch.2 at V+J, in particular, have a look at §2.2.3 and the impact of measurement on the ability to detect change or difference.

leave the issue of assessing and quantifying measurement precision and talk about statistics and improving measurement accuracy - $\$1.6$ of Notes!

application of polynomial regression to calibration

standard polynomial regression model

$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_k x^k + \epsilon$

$N(0, \sigma^2)$

$\beta$'s and $\sigma^2$ unknown parameters

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are typically used to do inference for unknown parameters —

Calibration Experiment: measure several items with both a "perfect" measurement system and my current/local system of interest - (this could be "get standard specimens from NIST and measure locally")

$x = \text{truth} = \text{known/constant}$

$y = \text{local measurement}$

random

standard linear model's theory tells us to do least squares, get fitted coefficient $b_0, b_1, \ldots, b_k$ and

$\hat{y} = g(x) = b_0 + b_1 x + \ldots + b_k x^k$

in the present context this suggests that upon measuring $y$ (locally) I should convert to $x$ via

$g^{-1}(y)$
Recall that you are taught how to make prediction limits in standard methods courses -

\[ \hat{y} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} \] 

Wrong in notes:

\[ \hat{y} \pm \text{std error} \hat{y} \hat{y} \]

should be

\[ \hat{y} \pm \sqrt{\text{std error}^2 + \text{std error}^2} \]

E.g. SLR version is

\[ (\hat{y} \pm \text{std error}) = \pm t_{1, n} \frac{\sigma}{\sqrt{n}} \sqrt{1 + \frac{(\bar{x} - \bar{x})^2}{(\bar{y} - \bar{y})^2}} \]

and I can "invert" prediction limits for \( x \) to get confidence sets for \( x \) - in nice cases (SLR version)

\[ \hat{y} = \beta_0 + \beta_1 x \]

\[ \hat{y} = b_0 + b_1 x \]

prediction limits for \( y \)

Confidence set for \( x \)

This works pretty well in practice as long as \( n \) is large and \( \sigma \) is small
Another set of issues regarding statistics and measurement is related to the question:

"What do I do about 'lack of precision' in measurement brought on by relatively crude gaging?"

Fact: typical normal theory formulas implicitly suppose that numbers put into them are "exact" in the sense that when I write $1.213$ inches I mean $1.213000000$ inches.

On the other hand, if $\sigma$ is small, the standard confidence guarantees are void.

Vardeman's advice when gaging is crude relative to variation in quantity of interest:

Don't pretend discrete things are continuous; treat them as discrete when:

1) generating or thinking about inappropriate values of statistics of interest
2) estimating parameters of underlying continuous random variables seen only through crude gaging.

$\sigma$ is big relative to the finest graduation in measurement $\rightarrow$ no problem.
Discrete Observations and Distributions of Common Statistics

Suppose $Y$ takes integer values, and let
\[ f(y) = P[Y = y] \]
be the pmf for $Y$ - an individual measured value.

From whence comes $f$?

One possibility is sampling under a fixed set of conditions and using an empirical r.f. den for $f$. 

<table>
<thead>
<tr>
<th>$y$</th>
<th>$f(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>0.55</td>
</tr>
<tr>
<td>1</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
</tr>
</tbody>
</table>

- A probability desn for $\bar{Y}$, $\sum_i Y_i$, $S$, $R$, etc.

One simple-minded way to approximate distributions of such statistics is through simulation.

For the sample mean, $\bar{Y}$:

Averages are essentially sums and desns of sums come easily by adding on diagonals of joint probability tables.

```
  | 0 | 1 | 2 |
---|---|---|---|
-1 | .04| .04| .01| .2
 0 | .01| .0275| .55|
 1 | .01| .04| .01| .2
 2 | .01| .0275| .01| .05|
```

- for some standard statistics it is easy to see how to write algorithms to compute exact desns.
Minitab Simulation of the Distributions of $\bar{x}, \bar{x}, s$ and $R$

MTB > Print C1 C2.

Data Display

<table>
<thead>
<tr>
<th>Row</th>
<th>C1</th>
<th>C2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.55</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.20</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.05</td>
</tr>
</tbody>
</table>

MTB > Randn 10000 c3-c7;
SUBC> Discrete C1 C2.

MTB > RMean c3-c7 c8.
MTB > RMedian c3-c7 c9.
MTB > RStDev c3-c7 c10.
MTB > RRange c3-c7 c11.

MTB > Describe 'Xbar' 'Median' 's' 'R'.

Descriptive Statistics: Xbar, Median, s, R

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>Median</th>
<th>TrimMean</th>
<th>StDev</th>
<th>SE Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xbar</td>
<td>10000</td>
<td>0.09604</td>
<td>0.00000</td>
<td>0.09903</td>
<td>0.34342</td>
<td>0.00343</td>
</tr>
<tr>
<td>Median</td>
<td>10000</td>
<td>0.04430</td>
<td>0.00000</td>
<td>0.04822</td>
<td>0.40129</td>
<td>0.00402</td>
</tr>
<tr>
<td>s</td>
<td>10000</td>
<td>0.70447</td>
<td>0.70711</td>
<td>0.71129</td>
<td>0.28972</td>
<td>0.00290</td>
</tr>
<tr>
<td>R</td>
<td>10000</td>
<td>1.6457</td>
<td>2.00000</td>
<td>1.6619</td>
<td>0.76662</td>
<td>0.0077</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Q1</th>
<th>Q3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xbar</td>
<td>-1.00000</td>
<td>1.40000</td>
<td>-0.20000</td>
<td>0.40000</td>
</tr>
<tr>
<td>Median</td>
<td>-1.00000</td>
<td>2.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>s</td>
<td>0.00000</td>
<td>1.64317</td>
<td>0.64772</td>
<td>0.89442</td>
</tr>
<tr>
<td>R</td>
<td>0.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>2.00000</td>
</tr>
</tbody>
</table>
So the pmf of $Y$ (n=2) is

<table>
<thead>
<tr>
<th>$y$</th>
<th>$P(Y=y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.04</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.22</td>
</tr>
<tr>
<td>0</td>
<td>0.325</td>
</tr>
<tr>
<td>0.5</td>
<td>0.24</td>
</tr>
<tr>
<td>1</td>
<td>0.055</td>
</tr>
<tr>
<td>1.5</td>
<td>0.02</td>
</tr>
<tr>
<td>2</td>
<td>0.0025</td>
</tr>
</tbody>
</table>

Den of $R$ (pages 12-13 of Notes):

WOLOG suppose $f(y)=0$ unless $1 \leq y \leq M$

Let

$$S_{kj} = \sum_{y=1}^{M} f(y) = P[k \leq Y \leq j]$$

$$0 \quad \text{otherwise}$$

compute + store for $1 \leq k, j \leq M$

Then for $Y_1, Y_2, \ldots, Y_n$ iid with marginal dist f let

$$A_{kj} = \{ k \leq \text{each } Y_i \leq j \}$$

$$B_{kj} = \{ \min Y_i = k \text{ and } \max Y_i = j \}$$

$$A_{k+1,j} \cap A_{kj}$$

$$A_{k+1,j}$$

$$A_{kj-1}$$

$$P(A_{kj}) = S_{kj}^n$$

$$M_{kj} = P(B_{kj}) = S_{kj}^n - S_{k+1,j}^n - S_{kj-1}^n + S_{k+1,j-1}^n$$

$$P[R=r] = \sum_{k=1}^{M} M_{kj}$$

largest possible value of $Y$
Testing/Confidence Gauges and the Estimation of parameters of an underlying cont.

(see Johnson Lee's paper on my Web page)

Suppose "really" $X$ is normal $(M, \sigma^2)$ but we only observe $X$ rounded to the nearest integer $X^*$.

You need to be very careful how you think here... lots of things people naively expect don't hold true.

How to go from a sample of $X^*$'s to estimates for $M, \sigma$? (sample mean and sample std deviation may not be such a good way to estimate parameters)

Use the "likelihood function" idea - this is essentially to treat

$$P \left( \text{data} \mid M, \sigma \right) = L(M, \sigma)$$

as a (random) function of parameters yielding inference about parameters.

E.g. Suppose $\eta$ is small

This will produce samples of $X$'s near all all 1's

This will produce a sample of $X^*$'s that is a binomial mixture of 1's and 2's.

$$L(M, \sigma) = \prod_{i=1}^{n} \left[ \Phi \left( \frac{x^*_i - 5 - \mu}{\frac{1}{2}} \right) - \Phi \left( \frac{x^*_i - 5 - \mu}{\frac{1}{2}} \right) \right]$$

$$\eta$$

12.5

13.5

13.0