531 Exam I / Spring 2001 / Key

1. This sample has range 1 in coded units. (The coded sample is $\{1,2,2,2,1,2,1,2\}$.) So we'll use Tables 1.3 and 1.6 of the notes.

13pts a) In the notation of page 16 of notes, $x^* = 1$, $n_{x^*} = 3$, $n_{x^*+1} = 5$, $m = 5$. For $n = 8$ and $m = 5$ Table 1.3 shows that $\Delta_1 = .527$ and $\Delta_2 = .281$.

Then $\Delta_L = .281$ and $\Delta_U = .527$

and in coded units the interval is

$$(1.5 - .281,\ 1.5 + .527)$$
$$(1.219\ ,\ 2.027)$$

In original units this is

$$(1.001219\ ,\ 1.002027)$$ inches

12pts b) The interval for σ based on a sample with range 1 is one-sided. Consulting Table 1.6, the upper endpoint is (in coded units)

$$\Lambda_{1,m} = .952$$

So the interval in original units is

$$(0,\ .000952)$$ inch

2. $P\left[\,|\bar{x} - \mu| > 2.88\sigma_{\bar{x}}\right] = P\left[\,|Z| > 2.88\right] = 2(1 - .9980)$

$= .004$

So the Shewhart all-OK ARL is $\frac{1}{.004} = 250$

13pts a) $k_1 = \frac{1}{2}(10 + 10.5208) = 10.2604$

$k_2 = \frac{1}{2}(10 + 9.4792) = 9.7396$

So $\mathcal{K} = \dfrac{10.2604 - 10}{(.5)/2.88} = 1.5$ and consulting Table 4.6, we see we want $\mathcal{H} = 1.47$ and then

$$h = 1.47\left(\frac{.5}{2.88}\right) = .2552$$

with $U_0 = L_0 = 0$

12pts b) If the current process mean is 10.5208 (and $\sigma_{\bar{x}} = .5/2.88$)

$$P[\bar{x} \text{ plots outside Shewhart limits}] = P\left[Z > \frac{10.5 - 10.1302}{.5/2.88}\right] + P\left[Z < \frac{9.5 - 10.1302}{.5/2.88}\right]$$

$$= P[Z > 2.13] + P[Z < -3.63]$$

$$\approx .0166$$

So Shewhart ARL is about $\frac{1}{.0166} \approx 60$

For the CUSUM, (using 4.15), (4.19) and (4.18)

$$\mathcal{H}^* = \frac{h}{\sigma_{\bar{x}}} = \frac{1.47\left(\frac{.5}{2.88}\right)}{\left(\frac{.5}{2.88}\right)} = 1.47$$

$$\mathcal{K}^* = \frac{10.2604 - 9.7396}{2\left(\frac{.5}{2.88}\right)} = 1.5$$

$$\mathcal{D}^* = \frac{|10.1302 - 10|}{\left(\frac{.5}{2.88}\right)} = .75$$

ARL is around (less than) 42 From Table 4.5

10 pts 3. a) Reading from the graph the values of x that produce prediction limits at 500, the 95% interval is about

$$(490, 545)$$

10 pts b) i) $UCL_{\chi^2} = p + 3\sqrt{2p} = 2 + 3\sqrt{4} = 8$

Roughly, we expect $Ee = 0$ and

$\mathrm{Var}\, e = \sigma^2$ ← from SLR model

(this ignores the fact that b's aren't β's). The MSE from SLR estimates σ^2. So sensible 3-sigma limits for e's are

$$UCL_e = 0 + 3\sqrt{MSE} = 3\sqrt{106.6} = 31$$

$$LCL_e = -3\sqrt{MSE} = -31$$

5 pts ii) I'd probably monitor e. e's will be large (in magnitude) only if the relationship between x and y changes. χ^2 could also be large by virtue of a process change (producing a change in the dsn of x's).

4. a) Using the basic measurement model, the "right" $\sigma_{measurement}$

11 pts is $\sigma_{repeatability}$ and

$$\sigma_y^2 = \sigma_x^2 + \sigma_{measurement}^2$$

so $\sigma_x = \sqrt{\sigma_y^2 - \sigma_{measurement}^2}$ = part-to-part standard deviation

Further, an obvious estimate is

$$\hat{\sigma}_x = \sqrt{\left(\frac{\bar{R}}{d_2(5)}\right)^2 - \left(\hat{\sigma}_{repeatability}\right)^2}$$

$$= \sqrt{\left(\frac{.010}{2.326}\right)^2 - (.002)^2} = .0038 \text{ inch}$$

b) 14pts $\hat{\sigma}_y = \dfrac{\bar{R}}{2.326}$ and $\hat{\sigma}_{\text{repeatability}}$ are plausibly modeled as independent. So the delta method gives

$$\operatorname{Var}\hat{\sigma}_x \approx \left(\frac{1}{2\sigma_x}\cdot 2\sigma_y\right)^2 \operatorname{Var}\hat{\sigma}_y$$

$$+\left(\frac{1}{2\sigma_x}(-2)\sigma_{\text{repeatability}}\right)^2 \operatorname{Var}\hat{\sigma}_{\text{repeatability}}$$

$$=\left(\frac{\sigma_y}{\sigma_x}\right)^2\left(\frac{1}{2.326}\right)^2\left(\frac{1}{40}\operatorname{Var}R\right)$$

from one sample

from $I \times J$ cells

$$+\left(\frac{\sigma_{\text{repeatability}}}{\sigma_x}\right)^2 \operatorname{Var}\left(\frac{\bar{R}}{d_2(3)}\right)$$

$$=\left(\frac{\sigma_y}{\sigma_x}\right)^2\left(\frac{1}{2.326}\right)^2\left(\frac{1}{40}\right)\left(d_3(5)\,\sigma_y\right)^2$$

$$+\left(\frac{\sigma_{\text{repeatability}}}{\sigma_x}\right)^2\left(\frac{1}{1.693}\right)^2\left(\frac{1}{30}\right)\left(d_3(3)\,\sigma_{\text{repeatability}}\right)^2$$

$$=\frac{1}{\sigma_x^2}\left[\sigma_y^4\left(\frac{.864}{2.326}\right)^2\frac{1}{40}+\sigma_{\text{repeatability}}^4\left(\frac{.888}{1.693}\right)^2\frac{1}{30}\right]$$

So

$$\sqrt{\operatorname{Var}\hat{\sigma}_x} = \frac{1}{\sigma_x}\sqrt{\sigma_y^4\left(\frac{.864}{2.326}\right)^2\frac{1}{40}+\sigma_{\text{repeatability}}^4\left(\frac{.888}{1.693}\right)^2\frac{1}{30}}$$

To produce a standard error, replace the σ's with estimates, namely

$$\hat{\sigma}_x = .0038$$

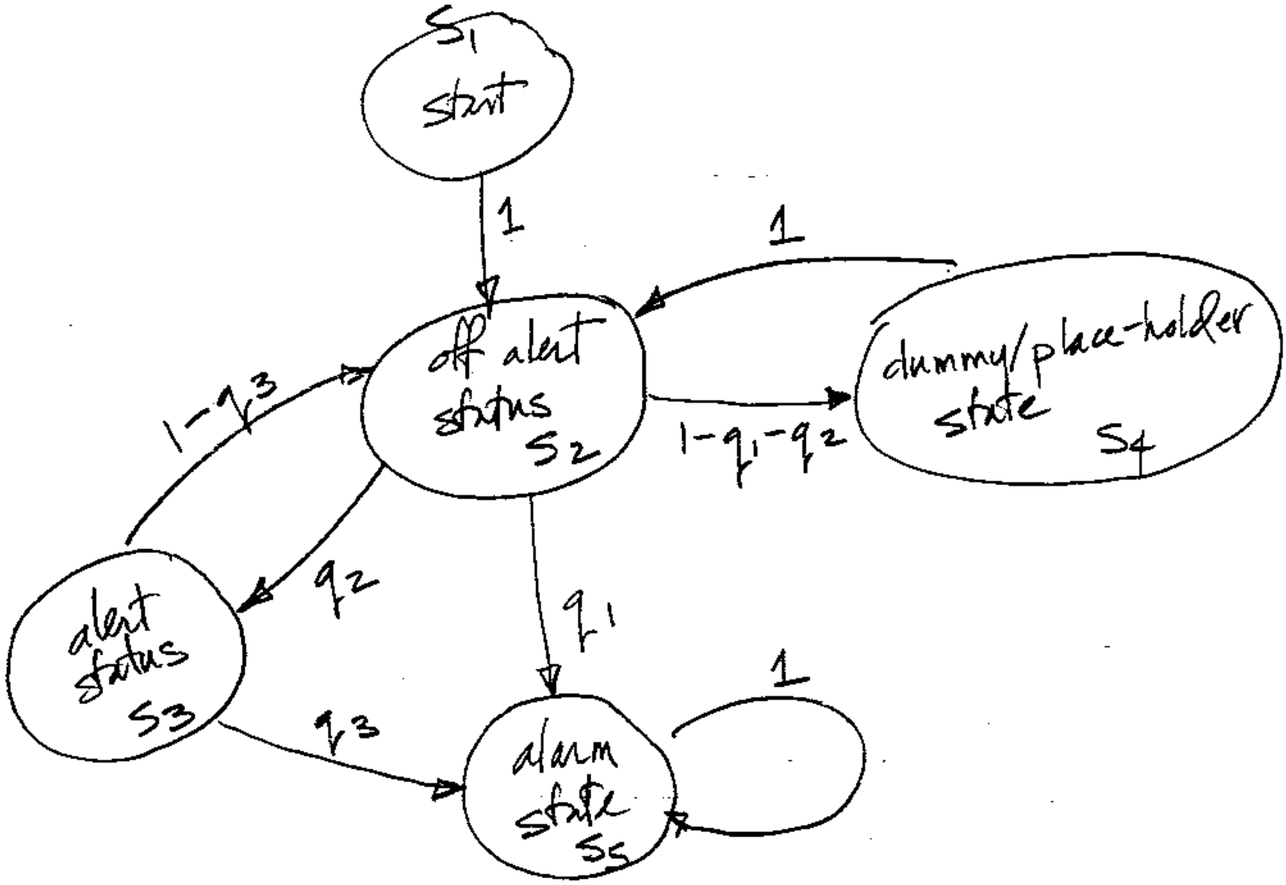
$$\hat{\sigma}_{\text{repeatability}} = .002$$

and

$$\hat{\sigma}_y = \frac{.010}{2.326} = .0043$$

Arithmetic then gives a standard error of .0003 inch

5. As suggested, consider half-hour periods and the MC with transition diagram below



Then the mean number of steps to absorption from S_1 (or S_4) is the mean number of half hours required to produce an alarm. That is, with

$$\underset{5\times5}{P} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & q_2 & 1-q_1-q_2 & q_1 \\ 0 & 1-q_3 & 0 & 0 & q_3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then with the usual $L = (I-R)^{-1}\mathbf{1}$, the mean clock time to alarm is

after some work

$$\frac{1}{2}L_1 = \frac{1}{2}L_4 = \frac{2+q_2q_3}{2(q_1+q_2q_3)}$$