Stat 511 Lecture 27

Note Title 3/26/2008

Recall BLUP for $\underset{\sim}{u}$: $\hat{u} = GZ'PY$

(and approximation for it $\hat{u} = \hat{G}Z'\hat{P}Y$)

More on the sense in which $\hat{u}$ is "best" – among all linear predictors it "minimizes" the covariance matrix for

$\underset{\sim}{u} - \underset{\sim}{\hat{u}}$ — an arbitrary linear predictor

over choices of $\underset{\sim}{\hat{u}}$ with mean $\underset{\sim}{0}$ – i.e. the BLUP makes small

$\mathrm{Var}(\underset{\sim}{u} - \underset{\sim}{\hat{u}})$

BTW for the BLUP

$$\mathrm{Var}(\underset{\sim}{u} - \underset{\sim}{\hat{u}}) = G - GZ'PZG$$

Both $\underset{\sim}{\hat{u}}$ and $\mathrm{Var}(\underset{\sim}{u} - \underset{\sim}{\hat{u}})$ depend upon $\underset{\sim}{\sigma}^2$ and are therefore not realizeable/not available for use – but, if I can estimate $\underset{\sim}{\sigma}^2$ (via ML or REML) I can get $\underset{\sim}{\hat{\hat{u}}}$ and hope that placing hats on G and P up there will get me something I can use to approximate/estimate $\mathrm{Var}(\underset{\sim}{u} - \hat{\hat{\underset{\sim}{u}}})$ i.e.

$$\widehat{\mathrm{Var}(\underset{\sim}{u} - \hat{\underset{\sim}{u}})} = \hat{G} - \hat{G}Z'\hat{P}Z\hat{G}$$

I hope $\longrightarrow$?? $\mathrm{Var}(\underset{\sim}{u} - \hat{\hat{\underset{\sim}{u}}})$

Further, it is possible to think about prediction of quantities like

$$\underbrace{\ell = \underbrace{\underset{\sim}{c}'}_{1\times k}\underbrace{\underset{\sim}{\beta}}_{k\times 1} + \underbrace{\underset{\sim}{s}'}_{1\times q}\underbrace{\underset{\sim}{u}}_{q\times 1}}$$

a linear combination of fixed and random effects

(this has the obvious generalization to $\underset{\sim}{\ell} = C\underset{\sim}{\beta} + S\underset{\sim}{u}$)

This generalizes both estimation of $\underset{\sim}{c}'\underset{\sim}{\beta}$ and prediction of elements of $\underset{\sim}{u}$

As it turns out, provided $\underset{\sim}{c}'\underset{\sim}{\beta}$ is estimable $(\underset{\sim}{c}' = \underset{\sim}{a}'X)$ the BLUP of ℓ is

$$\hat{\ell} = \underset{\sim}{a}' \hat{Y}^*(\underset{\sim}{\sigma}^2) + \underset{\sim}{s}'\hat{\underset{\sim}{u}}$$

BLUE of EY

BLUP for $\underset{\sim}{u}$

and a prediction variance for $\hat{\ell}$ is

$$\mathrm{Var}(\hat{\ell}-\ell) = \underset{\sim}{c}'(X'V^{-1}X)^{-}\underset{\sim}{c} + \underset{\sim}{s}'GZ'PG\underset{\sim}{s} - 2\underset{\sim}{a}'BZG\underset{\sim}{s}$$

Both $\hat{\ell}$ and $\mathrm{Var}(\hat{\ell}-\ell)$ depend upon unknown $\underset{\sim}{\sigma}^2$ and are thus not realizeable ... in the "obvious" way we'll use

$$\hat{\hat{\ell}} = \underset{\sim}{a}' \hat{Y}^*(\hat{\underset{\sim}{\sigma}}^2) + \underset{\sim}{s}'\hat{\hat{\underset{\sim}{u}}}$$

and I'll try to get an empirical approximation to $\mathrm{Var}(\hat{\hat{\ell}}-\ell)$ using

$$\widehat{\mathrm{Var}(\hat{\ell}-\ell)}$$

(this means plugging $\hat{\underset{\sim}{\sigma}}^2$ in for $\underset{\sim}{\sigma}^2$ in the earlier formula for $\mathrm{Var}(\hat{\ell}-\ell)$)

We also need some kind of measures of uncertainty/precision for our estimates of elements of $\underset{\sim}{\sigma}^2$ — in theory, one could apply theory about "the shape of the likelihood" (inversion of LRT's) (apply to restricted likelihood) — apparently that is really hard in practice — instead, state of the art software seems to rely on large sample theory for MLEs — (see again Section 7.3.1 of typed outline for a version of this) — recall

$\underset{\sim}{\theta}$ some r-dimensional vector of parameters and

$\ell(\underset{\sim}{\theta})$

is a log likelihood based on a "large sample" and

$\hat{\underset{\sim}{\theta}}$ is a maximizer of $\ell(\underset{\sim}{\theta})$

Standard large sample theory then suggests that $\mathrm{Var}(\hat{\underset{\sim}{\theta}})$ can be approximated/estimated using 2nd derivatives of $\ell(\underset{\sim}{\theta})$ at the MLE $\hat{\underset{\sim}{\theta}}$ i.e.

$$\widehat{\mathrm{Var}\,\hat{\underset{\sim}{\theta}}} = \left(- \frac{\partial^2}{\partial\theta_i\,\partial\theta_j} \ell(\underset{\sim}{\theta}) \Big|_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}} \right)^{-1}$$

so, for example, to set approximate confidence limits for θ_1

$$\hat{\theta}_1 \pm z \sqrt{\text{1st diagonal entry of that "inverse negative Hessian matrix"}}$$