Stat 511 Lecture 13

Note Title 2/13/2008

More on the Normal version of the Gauss-Markov LM...

Testing Consider a testable hypothesis

$$H_0 : C\beta = \underset{\sim}{d}$$

in the normal Gauss-Markov model – $\exists$ A such that $C = AX$ and

$$\widehat{C\beta}_{OLS} = A\hat{Y}$$

since $\hat{Y}$ is independent of SSE, so also is $\widehat{C\beta}_{OLS}$

and we can think of building a test of $H_0 : C\beta = \underset{\sim}{d}$ on

$$\widehat{C\beta}_{OLS} - \underset{\sim}{d} \quad \text{and} \quad SSE$$

$$MVN_\ell \left(C\beta - \underset{\sim}{d} , \sigma^2 C (X'X)^- C' \right)$$

Consider

$$\left(\widehat{C\beta}_{OLS} - \underset{\sim}{d}\right)' \left(\sigma^2 C (X'X)^- C'\right)^{-1} \left(\widehat{C\beta}_{OLS} - \underset{\sim}{d}\right)$$

as a measure of apparent mismatch between the data (represented by $\widehat{C\underset{\sim}{\beta}}_{OLS}$) and the hypothesis (represented by $\underset{\sim}{d}$) — this a generalization of the squared length of $\widehat{C\underset{\sim}{\beta}}_{OLS} - \underset{\sim}{d}$ which H_0 says has mean $\underset{\sim}{0}$ —

Notice that

$$\left(\sigma^2 C(X'X)^- C'\right)^{-1}\left(\sigma^2 C(X'X)^- C'\right) = I$$

is idempotent, so by Koehler's Thm 4.7

$$\frac{1}{\sigma^2}\left(\widehat{C\underset{\sim}{\beta}}_{OL} - \underset{\sim}{d}\right)'\left(C(X'X)^- C'\right)^{-1}\left(\widehat{C\underset{\sim}{\beta}}_{OLS} - \underset{\sim}{d}\right) \sim \chi^2_{\ell}(\delta^2)$$

SS_{H_0}

for $\delta^2 = \frac{1}{\sigma^2}\left(C\underset{\sim}{\beta} - \underset{\sim}{d}\right)'\left(C(X'X)^- C'\right)^{-1}\left(C\underset{\sim}{\beta} - \underset{\sim}{d}\right)$

(for cases where $\underset{\sim}{d} = \underset{\sim}{0}$ and we're talking about testing a hypothesis that $E\underset{\sim}{Y}$ is in some subspace of $C(X)$ (it's not easy to show but true that) this is a difference in "Full model" and "Reduced model" "regression sums of squares" — see "handout" for this Lecture)

When H_0 is true $\delta^2 = 0$ and when it is not true, $\delta^2 > 0$ and SS_{H_0} tends to be bigger than it would be with 0 noncentrality parameter — we already know that

$$\frac{SSE}{\sigma^2} \sim \chi^2_{n-\text{rank}(X)}$$

so some comparison of $\frac{SS_{H_0}}{\sigma^2}$ to $\frac{SSE}{\sigma^2}$ seems plausible as a way of testing $H: C\beta = \underset{\sim}{d}$

Def If $U \sim \chi^2_{\nu_1}$ independent of $V \sim \chi^2_{\nu_2}$ then the distribution of

$$\frac{U/\nu_1}{V/\nu_2}$$

is called the (Snedecor) F dsn with d.f. ν_1, ν_2

Def If $U \sim \chi^2_{\nu_1}(\lambda)$ independent of $V \sim \chi^2_{\nu_2}$ the dsn of

$$\frac{U/\nu_1}{V/\nu_2}$$

is called the noncentral F dsn with d.f. ν_1 and ν_2 and noncentrality parameter λ

like the χ^2 dsns the noncentral F dsns are "pulled right" relative to their central F counterparts

Define

$$F = \frac{\frac{1}{\sigma^2} SS_{H_0} / l}{\frac{1}{\sigma^2} SSE / (n - \text{rank}(X))} = \frac{MS_{H_0}}{MSE}$$

and conclude that F is $F_{l,\, n-\text{rank}(X)}$ with noncentrality parameter $\lambda = \delta^2$ ← from before

So, an α-level test of $H_0: \underset{\sim}{C}\underset{\sim}{\beta} = \underset{\sim}{d}$ can be had by

rejecting H_0 if $F >$ upper α pt of $F_{\nu_1, \nu_2}(0)$

or a p-value for this hypothesis is

$\left(\text{central } F_{\nu_1,\nu_2} \text{ probability}\right)$ to right of $\left(\text{observed value of F statistic}\right)$

Further I can use the non-central version of F dsn to to evaluate the "power" of this test – i.e. the noncentral F probability to the right of (central F) cut-off power = P[test rejects H_0]

$= P\left[\text{a noncentral } F_{\ell, n-\text{rank}(X)}(\delta^2) > \text{upper } \alpha \text{ pt of the central } F_{\ell, n-\text{rank}(X)} \text{ dsn}\right]$

for $\delta^2 = \frac{1}{\sigma^2}(C\beta - d)'(C(X'X)^- C')^{-1}(C\beta - d)$

Cartoon