Stat 511 Lecture 5 "Handout"

The following are equivalent regarding $\mathbf{c} \in \Re^k$

1) $\exists\, \mathbf{a} \in \Re^n$ such that $\mathbf{a}'\mathbf{X}\boldsymbol{\beta} = \mathbf{c}'\boldsymbol{\beta}\ \forall \boldsymbol{\beta}$
2) $\mathbf{c} \in C(\mathbf{X}')$ ($\mathbf{c}'$ is a linear combination of the rows of $\mathbf{X}$)
3) $\mathbf{X}\boldsymbol{\beta}_1 = \mathbf{X}\boldsymbol{\beta}_2 \ \Rightarrow\ \mathbf{c}'\boldsymbol{\beta}_1 = \mathbf{c}'\boldsymbol{\beta}_2$

Condition 3) is a condition about the lack of ambiguity of the value of $\mathbf{c}'\boldsymbol{\beta}$. Condition 1) is

$$\mathbf{c}'\boldsymbol{\beta} = \mathbf{a}'\mathbf{X}\boldsymbol{\beta} = \mathbf{a}'\mathrm{E}\mathbf{Y} = \mathrm{E}\mathbf{a}'\mathbf{Y}$$

so that $\mathbf{a}'\mathbf{Y}$ can be used to estimate $\mathbf{c}'\boldsymbol{\beta}$ in an unbiased fashion.

Proof: First suppose that 2) holds. $\mathbf{c} \in C(\mathbf{X}') \ \Rightarrow\ \exists\, \mathbf{a}$ such that $\mathbf{c}' = \mathbf{a}\mathbf{X}$. So for this $\mathbf{a}$, $\mathbf{c}'\boldsymbol{\beta} = \mathbf{a}'\mathbf{X}\boldsymbol{\beta}\ \forall \boldsymbol{\beta}$ and 1) holds.

Next suppose that 1) holds, i.e. that $\exists\, \mathbf{a} \in \Re^n$ such that $\mathbf{a}'\mathbf{X}\boldsymbol{\beta} = \mathbf{c}'\boldsymbol{\beta}\ \forall \boldsymbol{\beta}$. Suppose $\mathbf{X}\boldsymbol{\beta}_1 = \mathbf{X}\boldsymbol{\beta}_2$. For this $\mathbf{a}$,

$$\mathbf{c}'\boldsymbol{\beta}_1 = \mathbf{a}'\mathbf{X}\boldsymbol{\beta}_1 = \mathbf{a}'\mathbf{X}\boldsymbol{\beta}_2 = \mathbf{c}'\boldsymbol{\beta}_2$$

and 3) holds.

Finally, suppose that 3) holds. It is "obviously" equivalent to write

$$\mathbf{X}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2) = \mathbf{0} \ \Rightarrow\ \mathbf{c}'(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2) = \mathbf{0}\ \forall\ \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$$

That is, it is equivalent to 3) to write

$$\mathbf{X}\mathbf{d} = \mathbf{0} \ \Rightarrow\ \mathbf{c}'\mathbf{d} = \mathbf{0}\ \forall\ \mathbf{d}$$

Then, the claim that 3)$\Rightarrow$2) is the claim that

$$\left\{\mathbf{c} \,|\, \left[\mathbf{X}\mathbf{d} = \mathbf{0} \ \Rightarrow\ \mathbf{c}'\mathbf{d} = \mathbf{0}\ \forall\ \mathbf{d}\right] \text{ holds}\right\} \subset C(\mathbf{X}')$$

Suppose that $\mathbf{c}$ is such that $\left[\mathbf{X}\mathbf{d} = \mathbf{0} \ \Rightarrow\ \mathbf{c}'\mathbf{d} = \mathbf{0}\ \forall\ \mathbf{d}\right]$ holds. Write

$$\mathbf{c} = \mathbf{P}_{\mathbf{X}'}\mathbf{c} + (\mathbf{I} - \mathbf{P}_{\mathbf{X}'})\mathbf{c}$$

and let $\mathbf{d}^* = (\mathbf{I} - \mathbf{P}_{\mathbf{X}'})\mathbf{c}$. We can then argue that $\mathbf{d}^* = \mathbf{0}$ as follows. Clearly, $\mathbf{d}^* \in C(\mathbf{X}')^{\perp}$ so it must be that $\mathbf{c}'\mathbf{d}^* = \mathbf{0}$ from the condition $[\ \ ]$. But

$$\mathbf{c}'\mathbf{d}^* = \mathbf{c}'(\mathbf{I} - \mathbf{P}_{\mathbf{x}'})\mathbf{c} = \mathbf{c}'\mathbf{c} - \mathbf{c}'\mathbf{P}_{\mathbf{x}'}\mathbf{c}$$

so that $\mathbf{c}'\mathbf{c} = \mathbf{c}'\mathbf{P}_{\mathbf{x}'}\mathbf{c}$. But

$$\begin{aligned}
\mathbf{c}'\mathbf{c} &= \mathbf{c}'\left(\mathbf{P}_{\mathbf{X}'} + (\mathbf{I} - \mathbf{P}_{\mathbf{X}'})\right)\mathbf{c} \\
&= \mathbf{c}'\mathbf{P}_{\mathbf{X}'}\mathbf{c} + \mathbf{c}'(\mathbf{I} - \mathbf{P}_{\mathbf{X}'})\mathbf{c} \\
&= \mathbf{c}'\mathbf{P}_{\mathbf{X}'}\mathbf{c} + \mathbf{c}'(\mathbf{I} - \mathbf{P}_{\mathbf{X}'})'(\mathbf{I} - \mathbf{P}_{\mathbf{X}'})\mathbf{c}
\end{aligned}$$

Then since $\mathbf{c}'(\mathbf{I} - \mathbf{P}_{\mathbf{X}'})'(\mathbf{I} - \mathbf{P}_{\mathbf{X}'})\mathbf{c} = \mathbf{d}^{*\prime}\mathbf{d}^*$, we have $\mathbf{d}^{*\prime}\mathbf{d}^* = 0$ so that $\mathbf{d}^* = \mathbf{0}$. Thus $\mathbf{c} = \mathbf{P}_{\mathbf{X}'}\mathbf{c}$ so that $\mathbf{c} \in C(\mathbf{X}')$. □