# Stat 511 HW#6 Spring 2009

This assignment consists of problems on mixed linear models. Most are repeats from HW #10 of 2003 and HW#8 of 2004. All of these problems requiring computing should be done using both the `lme()` function in the `nlme` package in `R` (used in 2003 and 2004 versions of Stat 511) and the `lmer()` function in the `lme4` package used first in 2008. (See Chapter 8 of Faraway's *Extending the Linear Model with* `R` for examples of the use of the `lmer()` function.) The syntax of the new function is easier to understand and use than that of the earlier one, but the earlier package has more useable methods associated with it.

1. Below is a very small set of fake unbalanced 2-level nested data. Consider the analysis of these under the mixed effects model

$$y_{ijk} = \mu + \alpha_i + \beta_{ij} + \varepsilon_{ijk}$$

where the only fixed effect is $\mu$, the random effects are all independent with the $\alpha_i$ iid $\mathrm{N}\left(0,\sigma_\alpha^2\right)$, the $\beta_{ij}$ iid $\mathrm{N}\left(0,\sigma_\beta^2\right)$, and the $\varepsilon_{ijk}$ iid $\mathrm{N}\left(0,\sigma^2\right)$.

| **Level of A** | **Level of B within A** | **Response** |
|---|---|---|
| 1 | 1 | 6.0, 6.1 |
| | 2 | 8.6, 7.1, 6.5, 7.4 |
| 2 | 1 | 9.4, 9.9 |
| | 2 | 9.5, 7.5 |
| | 3 | 6.4, 9.1, 8.7 |

After loading the `MASS`, `nlme`, and `lme4` packages and specifying the use of the sum restrictions as per

```
> options(contrasts=c("contr.sum","contr.sum"))
```

use the `lme()` and `lmer()` functions to do an analysis of these data. (See Section 8.6 of Faraway for something parallel to this problem.)

a) Run `summary()` on the results of your `lme()` and `lmer()` calls. Then do what is necessary to get "95% intervals" here for both fixed effects and for the random effects or their ratios. In the first case, you may simply use the `intervals()` function to get approximate confidence intervals. In the second case, the current version of the `lmer()` function produces point estimates of variance components (and their square roots). One must do more to get a (Bayes credible) interval (presumably based on "Jeffreys priors") for the "error" standard deviation $\sigma$ in a mixed linear model (presented directly) *and multipliers (listed as elements of* `$ST` *) to be applied to those end points in order to produce intervals for the other model standard deviations*. (The elements of `$ST` then portray "relative standard deviations" or the ratios of the other model standard deviations to $\sigma$.) (Credible intervals are not quite confidence intervals but can be thought of as roughly comparable.) You may employ code like

```
> sims <- mcmcsamp(lmer.fit, 50000)
> HPDinterval(sims)
```

Then compute an exact confidence interval for $\sigma$ based on the mean square error (or pooled variance from the 5 samples of sizes 2,4,2,2, and 3). How do these limits compare to what `R` provides for the mixed model in this analysis?

b) A fixed effects analysis can be made here (treating the $\alpha_i$ and $\beta_{ij}$ as unknown fixed parameters). Do this using the `lm()` function. Run `summary()` and `confint()` on the result of your call. Note that the estimate of $\sigma$ produced by this analysis is exactly the one based on the mean square error in a).

c) Run `predict()` or `fitted()` (and `random.effects()` or `ranef()` for the mixed model analyses) on the results of the calls from parts b) and c) above. Identify the predictions from the fixed effects analysis as simple functions of the data values. (What are these predictions?) Notice that the predictions from the mixed effects analysis are substantially different from those based on the fixed effects model.

2. The article "Variability of Sliver Weights at Different Carding Stages and a Suggested Sampling Plan for Jute Processing" by A. Lahiri (*Journal of the Textile Institute*, 1990) concerns the partitioning of variability in "sliver weight." (A sliver is a continuous strand of loose, untwisted wool, cotton, etc., produced along the way to making yarn.) For a particular mill, 3 (of many) machines were studied, using 5 (10 mm) pieces of sliver cut from each of 5 rolls produced on the machines. The weights of the (75) pieces of sliver were determined and a standard hierarchical (balanced data) ANOVA table was produced as below. (The units of weight were not given in the original article.)

| **Source** | *SS* | *df* |
|---|---|---|
| Machines | 1966 | 2 |
| Rolls | 644 | 12 |
| Pieces | 280 | 60 |
| Total | 2890 | 74 |

Use the same mixed effects model as in Problem 1 and do the following.

a) Make 95% confidence intervals for each of the 3 standard deviations $\sigma_\alpha, \sigma_\beta$ and $\sigma$. Based on these, where do you judge the largest part of variation in measured weight to come from? (From differences between pieces for a given roll? From differences between rolls from a given machine? From differences between machines?) (Use Cochran- Satterthwaite for $\sigma_\alpha$ and $\sigma_\beta$ and give an exact interval for $\sigma$.)

b) Suppose for sake of illustration that the grand average of all 75 weight measurements was $\bar{y}_{...} = 35.0$. Use this and make a 95% confidence interval for the model parameter $\mu$.

3. Consider the fake unbalanced $3 \times 3$ factorial data of Problem 3 on HW 4. Here do a random effects analyses of those

$$y_{ijk} = k\text{th response at level } i \text{ of A and level } j \text{ of B}$$

based on a two-way random effects model without interaction

$$y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

where $\mu$ is an unknown constant, the $\alpha_i$ are iid $\mathrm{N}\left(0,\sigma_\alpha^2\right)$, the $\beta_j$ are iid $\mathrm{N}\left(0,\sigma_\beta^2\right)$, the $\varepsilon_{ij}$ are iid $\mathrm{N}\left(0,\sigma^2\right)$ and all the sets of random effects are all independent.
After loading the `MASS`, `nlme`, and `lme4` packages (and specifying the use of the sum restrictions), do an analysis of these data. (See Section 8.7 of Faraway for something parallel to this problem.)

Run the `summary()` function on the results of your `lme()` and `lmer()` calls. Make appropriate function calls to get predictions/fitted values and 95% intervals. What are 95% limits for the model parameters $\mu$ and $\sigma$? What do the intervals you are able to produce tell you about the relative sizes of the standard deviations $\sigma, \sigma_\alpha$, and $\sigma_\beta$?

4. The data set below is taken from page 54 of *Statistical Quality Assurance Methods for Engineers* by Vardeman and Jobe. It gives burst strength measurements made by 5 different technicians on small pieces of paper cut from 2 different large sheets (each technician tested 2 small pieces from both sheets).

| | | **Technician (B)** | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| **Sheet (A)** | 1 | 13.5, 14.8 | 10.5, 11.7 | 12.9, 12.0 | 8.8, 13.5 | 12.4, 16.0 |
| | 2 | 11.3, 12.0 | 14.0, 12.5 | 13.0, 13.1 | 12.6, 12.7 | 11.0, 10.6 |

Vardeman and Jobe present an ANOVA table for these data based on a two-way random effects model *with* interaction

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk}$$

(where $\mu$ is the only fixed effect).

| Source | *SS* | *df* | E *MS* |
|---|---|---|---|
| Sheet (A) | .5445 | 1 | $10\sigma_\alpha^2 + 2\sigma_{\alpha\beta}^2 + \sigma^2$ |
| Technician (B) | 2.692 | 4 | $4\sigma_\beta^2 + 2\sigma_{\alpha\beta}^2 + \sigma^2$ |
| Sheet*Technician (A*B) | 24.498 | 4 | $2\sigma_{\alpha\beta}^2 + \sigma^2$ |
| Error | 20.955 | 10 | $\sigma^2$ |
| Total | 48.6895 | 19 | |

a) Use `R` and an ordinary (fixed effects) linear model analysis to verify that the sums of squares above are correct.

b) In the context of a "gauge study" like this, $\sigma$ is usually called the "repeatability" standard deviation. Give a 95% exact confidence interval for $\sigma$ (based on the error mean square in the ANOVA table.)

c) Under the two-way random effects model above, the quantity $\sqrt{\sigma_\beta^2 + \sigma_{\alpha\beta}^2}$ is the long run standard deviation that would be seen in many technicians measure from the same sheet in the absence of any repeatability variation. It is called the "reproducibility" standard deviation. Find a linear combination of mean squares that has expected value the reproducibility variance, use Cochran-Satterthwaite and make approximate 95% confidence limits for that variance, and then convert those limits to limits for the reproducibility standard deviation here.

d) Now use `lme()` and `lmer()` in an analysis of these data. Run the `summary()` function on the objects that result. Run appropriate functions to find predictions/fitted values and 95% intervals. What are point estimates of the repeatability and reproducibility standard deviation based on REML? In light of the intervals concerning the standard deviations that `R` produces, how reliable do you think the point estimate of reproducibility standard deviation is? Does your approximate interval from c) really do justice to the uncertainty here? (In retrospect, it should be obvious that one can't learn much about technician-to-technician variability based on a sample of 5 technicians!)

(Syntax like `lmer(y ~ (1 | A) + (1| B) + (1| A:B))` should work here.)

5. Below is a small set of fake unbalanced 2-way factorial (in factors A and B) data from 2 (unbalanced) blocks.

| Level of A | Level of B | Block | Response |
|---|---|---|---|
| 1 | 1 | 1 | 9.5 |
| 1 | 2 | 1 | 11.3 |
| 1 | 3 | 1 | 8.4 |
| 1 | 3 | 1 | 8.5 |
| 2 | 1 | 1 | 6.7 |
| 2 | 2 | 1 | 9.4 |
| 2 | 3 | 1 | 6.1 |
| 1 | 1 | 2 | 12.0 |
| 1 | 1 | 2 | 12.2 |
| 1 | 2 | 2 | 13.1 |
| 1 | 3 | 2 | 10.7 |
| 2 | 1 | 2 | 8.9 |
| 2 | 2 | 2 | 10.8 |
| 2 | 3 | 2 | 7.4 |
| 2 | 3 | 2 | 8.6 |

Consider an analysis of these data based on the model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \tau_k + \varepsilon_{ijk} \qquad (*)$$

first where (*) specifies a fixed effects model (say under the sum restrictions) and then where the $\tau_k$ are iid $\mathrm{N}\left(0,\sigma_\tau^2\right)$ random block effects independent of the iid $\mathrm{N}\left(0,\sigma^2\right)$ random errors $\varepsilon_{ijk}$ .

Load the `MASS`, `nlme`, and `lme4` packages to your `R` session. (See Section 8.4 of Faraway for something parallel to this problem.)

a) Create vectors `A,B,Block,y` in `R`, corresponding to the columns of the data table above. Make from the first 3 of these objects that `R` can recognize as factors by typing

```
> A<-as.factor(A)
> B<-as.factor(B)
> FBlock<-as.factor(Block)
```

Tell `R` you wish to use the sum restrictions in linear model computations by typing

```
> options(contrasts=c("contr.sum","contr.sum"))
```

Then note that a fixed effects linear model with equation (*) can be fit and some summary statistics viewed in R by typing

```
> lmf.out<-lm(y~A*B+FBlock)
> summary(lmf.out)
```

What is an estimate of $\sigma$ in this model?

b) Now use `lme()` and `lmer()` in an analysis of these data under model (*). Run the function `summary()` and get intervals for the results of the calls you make.

In terms of inferences for the fixed effects, how much difference do you see between the outputs for the analysis with fixed block effects and the analysis with random block effects? How do the estimates of $\sigma$ compare for the two models? Compare a 95% confidence interval for $\sigma$ in the fixed effects model to what you get from these calls.

In the fixed effects analysis of a), one can estimate $\tau_2 = -\tau_1$. Is this estimate consistent with the estimate of $\sigma_\tau^2$ produced here? Explain.

c) What are the BLUE and (estimated) BLUP of $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_1$ in respectively fixed and mixed model analyses? Notice that in the random blocks analysis, it makes good sense to predict a new observation from levels 1 of A and 1 of B, *from an as yet unseen third block*. This would be $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_3$. What value would you use to predict this? Why does it not make sense to do predict a new observation with mean $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_3$ in the fixed effects context?

6. Consider the scenario of the example beginning on page 1190 of the 4$^{\text{th}}$ Edition of Neter and friends. This supposedly concerns sales of pairs of athletic shoes under two different advertising campaigns in several test markets. The data from Neter, et al. Table 29.10 are on the last page of this assignment.

Consider an analysis of these data based on the "split plot"/"repeated measures" mixed effects model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \gamma_{jk} + \varepsilon_{ijk}$$

for $y_{ijk}$ the response at time $i$ from the $k$th test market under ad campaign $j$, where the random effects are the $\gamma_{jk}$'s and the $\varepsilon_{ijk}$'s.

Use the `lme()` and `lmer()` functions to do an analysis of these data. (See Section 8.5 of Faraway for something parallel to this problem.)

a) Neter, and friends produce a special ANOVA table for these data. An ANOVA table with the same sums of squares and degrees of freedom can be obtained in R using fixed effects calculations by typing

```
> TIME<-as.factor(time)
> AD<-as.factor(ad)
> TEST<-as.factor(test)
```

```
> lmf1.out<-lm(y~1+TIME*(AD/TEST))
> anova(lmf1.out)
```

Identify the sums of squares and degrees of freedom in this table by the names used in class ( $SSA, SSB, SSAB, SSC(B), SSE$ ,etc.). Then run a fictitious 3-way full factorial analysis of these data by typing

```
> lmf2.out<-lm(y~1+TIME*AD*TEST)
> anova(lmf2.out)
```

Verify that the sums of squares and degrees of freedom in the first ANOVA table can be gotten from this second table. (Show how.)

b) Now use `lme()` and `lmer()` in an analysis of these data. Run the function `summary()` on the results of the calls you make. Get predictions/fitted values, predictions of random effects, and 95% intervals. How do $\sigma_\gamma$ and $\sigma$ compare here?

c) Make exact 95% confidence limits for $\sigma$ based on the chi-square distribution related to $MSE$ . Then use the Cochran-Satterthwaite approximation to make 95% limits for $\sigma_\gamma$ . How do these two sets of limits fit with what `R` produced in part b)?

d) Make 95% confidence limits for the difference in Time 1 and Time 2 main effects. Make 95% confidence limits for the difference in Ad Campaign 1 and Ad Campaign 2 main effects.

e) How would you predict the Time 2 sales result for Ad Campaign 2 in another (untested) market? (What value would you use?)

**Data from Neter, et al. Table 29.10**

| Time Period A | Ad Campaign B | Test Market C(B) | Response (Sales?) |
|---|---|---|---|
| 1 | 1 | 1 | 958 |
| 1 | 1 | 2 | 1,005 |
| 1 | 1 | 3 | 351 |
| 1 | 1 | 4 | 549 |
| 1 | 1 | 5 | 730 |
| 1 | 2 | 1 | 780 |
| 1 | 2 | 2 | 229 |
| 1 | 2 | 3 | 883 |
| 1 | 2 | 4 | 624 |
| 1 | 2 | 5 | 375 |
| 2 | 1 | 1 | 1,047 |
| 2 | 1 | 2 | 1,122 |
| 2 | 1 | 3 | 436 |
| 2 | 1 | 4 | 632 |
| 2 | 1 | 5 | 784 |
| 2 | 2 | 1 | 897 |
| 2 | 2 | 2 | 275 |
| 2 | 2 | 3 | 964 |
| 2 | 2 | 4 | 695 |
| 2 | 2 | 5 | 436 |
| 3 | 1 | 1 | 933 |
| 3 | 1 | 2 | 986 |
| 3 | 1 | 3 | 339 |
| 3 | 1 | 4 | 512 |
| 3 | 1 | 5 | 707 |
| 3 | 2 | 1 | 718 |
| 3 | 2 | 2 | 202 |
| 3 | 2 | 3 | 817 |
| 3 | 2 | 4 | 599 |
| 3 | 2 | 5 | 351 |