STAT 511 HW#2 SPRING 2009

PROBLEM 1:

a)

$$A = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & -1/4 \end{bmatrix}$$

b)

i. It is known that $EY = X\beta \in C(X)$. In addition $H_0: C\beta = 0 \equiv AX\beta = AEY = 0$ which means that A is orthogonal to EY, that is $EY \subset C(A')^{\perp}$. So $EY \subset C(X) \cap C(A')^{\perp}$.

ii. $C(X_0) \subset C(X)$ since columns of X_0 can be written as linear combinations of the columns of X

iii. Using R we can show that $P_{A'}X_0 = 0$, which means that $C(X_0) \subset C(A')^{\perp}$
PX is the same as the function named project that you were asked to write on HW1 problem 9.
```
A<-1/4*matrix(c(1,1,1,1,-1,-1,-1,-1,1,1,-1,-1,1,1,-1,-1),2,8,byrow=TRUE)
X0<-t(matrix(c(1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,1,1),2,8,byrow=TRUE))
```
PX(t(A))%*%X0 $=0_{8\times 2}$

iv. By ii. and iii. $C(X_0) \subset C(X) \cap C(A')^{\perp}$

v. $\dim(C(X_0)) = 2$ and $\dim((X) \cap C(A')^{\perp}) = 2$

vi. By iv. And v. $C(X_0) = C(X) \cap C(A')^{\perp}$

vii. Therefore by i. and vi. $EY \subset C(X_0)$

c) Test of main effects for α and β

PROBLEM 2:

a)

$$E[Y - \hat{Y}] = E[(I - P_X)Y] = (I - P_X)E[Y] = (I - P_X)X\beta = X\beta - X\beta = 0$$

$$Var[Y - \hat{Y}] = Var[(I - P_X)Y] = (I - P_X)Var[Y](I - P_X)' = (I - P_X)\sigma^2 I(I - P_X)' = \sigma^2(I - P_X)$$

$$Var\left(\begin{bmatrix} \hat{Y} \\ Y - \hat{Y} \end{bmatrix}\right) = Var\left(\begin{bmatrix} P_X \\ I - P_X \end{bmatrix} Y\right) = \begin{bmatrix} P_X \\ I - P_X \end{bmatrix} Var(Y)[P_X \quad I - P_X] = \begin{bmatrix} P_X \\ I - P_X \end{bmatrix}\sigma^2 I[P_X \quad I - P_X] =$$
$$\sigma^2 \begin{bmatrix} P_X & 0 \\ 0 & I - P_X \end{bmatrix}$$ by Appendix 7.1 1.

b)

$$\begin{aligned} E[(Y - \hat{Y})'(Y - \hat{Y})] &= E[((I - P_X)Y)'((I - P_X)Y)] = E[Y'(I - P_X)'(I - P_X)Y] = E[Y'(I - P_X)Y] \\ &= tr((I - P_X)\sigma^2 I) + (X\beta)'(I - P_X)X\beta = \sigma^2 tr(I - P_X) = \sigma^2 rank(I - P_X) \\ &= \sigma^2(n - rank(X)) \end{aligned}$$

PROBLEM 3:

Create matrices in R

```
X<-matrix(c(rep(1,9),rep(0,7),rep(1,2),rep(0,7) ,1,rep(0,7),rep(1,2)),7,5,byrow=FALSE)
Y<-matrix(c(3,1,4,6,3,5,4),7,1,byrow=TRUE)
C<-matrix(c(1,1,0,0,0,1,0,1,0,0,0,0,0,1,-1,1,.25,.25,.25,.25),4,5,byrow=TRUE)
V1<-diag(c(1,4,4,1,1,1,4))
V2<-V1
V2[2,1]<-V2[1,2]<-1
V2[3,4]<-V2[4,3]<--1
```

Create a function to compute $V^{-\frac{1}{2}}$

```
W<-function(X)
{
    U<-eigen(X)$vectors
    Dinv<-diag(1/sqrt(eigen(X)$values))
    return(U%*%Dinv%*%t(U))
}
```

Ordinary Least Squares Estimates

$$EY = P_X Y$$
$$C\beta = C(X'X)^- X'Y$$
$$CovY = P_X V P_X$$
$$CovC\beta = C(X'X)^- X'VX(X'X)^- C'$$

```
EY<-PX(X)%*%Y
CB<-C%*%ginv(t(X)%*%X)%*%t(X)%*%Y
EY
2.0    2.0    5.0    5.0    3.0    4.5    4.5
CB
2.0    5.0    -1.5    3.625
```

For V1

```
CovY1<-PX(X)%*%V1%*%PX(X)
CovCB1<- C%*%ginv(t(X)%*%X)%*%t(X)%*%V1%*%t(C%*%ginv(t(X)%*%X)%*%t(X))
CovY1
     [,1] [,2] [,3] [,4] [,5] [,6] [,7]
[1,] 1.25 1.25 0.00 0.00    0 0.00 0.00
[2,] 1.25 1.25 0.00 0.00    0 0.00 0.00
[3,] 0.00 0.00 1.25 1.25    0 0.00 0.00
[4,] 0.00 0.00 1.25 1.25    0 0.00 0.00
[5,] 0.00 0.00 0.00 0.00    1 0.00 0.00
[6,] 0.00 0.00 0.00 0.00    0 1.25 1.25
[7,] 0.00 0.00 0.00 0.00    0 1.25 1.25
```

```
CovCB1
       [,1]   [,2]    [,3]     [,4]
[1,] 1.2500 0.0000  0.0000  0.31250
[2,] 0.0000 1.2500  0.0000  0.31250
[3,] 0.0000 0.0000  2.2500 -0.06250
[4,] 0.3125 0.3125 -0.0625  0.29688
```

For V2

```
CovY2<-PX(X)%*%V2%*%PX(X)
CovCB2<- C%*%ginv(t(X)%*%X)%*%t(X)%*%V2%*%t(C%*%ginv(t(X)%*%X)%*%t(X))
CovY2
     [,1] [,2] [,3] [,4] [,5] [,6] [,7]
[1,] 1.75 1.75 0.00 0.00    0 0.00 0.00
[2,] 1.75 1.75 0.00 0.00    0 0.00 0.00
[3,] 0.00 0.00 0.75 0.75    0 0.00 0.00
[4,] 0.00 0.00 0.75 0.75    0 0.00 0.00
[5,] 0.00 0.00 0.00 0.00    1 0.00 0.00
[6,] 0.00 0.00 0.00 0.00    0 1.25 1.25
[7,] 0.00 0.00 0.00 0.00    0 1.25 1.25
CovCB2
       [,1]   [,2]    [,3]     [,4]
[1,] 1.7500 0.0000  0.0000  0.43750
[2,] 0.0000 0.7500  0.0000  0.18750
[3,] 0.0000 0.0000  2.2500 -0.06250
[4,] 0.4375 0.1875 -0.0625  0.29688
```

Generalized Least Squares Estimates

$$U = V^{-\frac{1}{2}}Y$$
$$W = V^{-\frac{1}{2}}X$$
$$EY = V^{\frac{1}{2}}P_W U$$
$$C\beta = C(W'W)^{-}W'U$$
$$CovY = V^{\frac{1}{2}}P_w V^{\frac{1}{2}\prime}$$
$$CovC\beta = C(W'W)^{-}C'$$

For V1

```
U1<-W(V1)%*%Y
W1<-W(V1)%*%X
EY1<-solve(W(V1))%*%PX(W1)%*%U1
CB1<-C%*%ginv(t(W1)%*%W1)%*%t(W1)%*%U1
CovY1<-solve(W(V1))%*%PX(W1)%*%t(solve(W(V1)))
CovCB1<-C%*%ginv(t(W1)%*%W1)%*%t(C)
```

```
EY1
2.6    2.6    5.6    5.6    3.0    4.8    4.8
CB1
2.6    5.6    -1.8    4.0
CovY1
   [,1] [,2] [,3] [,4] [,5] [,6] [,7]
   0.8  0.8  0.0  0.0    0  0.0  0.0
[2,]  0.8  0.8  0.0  0.0    0  0.0  0.0
[3,]  0.0  0.0  0.8  0.8    0  0.0  0.0
[4,]  0.0  0.0  0.8  0.8    0  0.0  0.0
[5,]  0.0  0.0  0.0  0.0    1  0.0  0.0
[6,]  0.0  0.0  0.0  0.0    0  0.8  0.8
[7,]  0.0  0.0  0.0  0.0    0  0.8  0.8
CovCB1
     [,1] [,2] [,3]   [,4]
[1,]  0.8  0.0 0.00 0.2000
[2,]  0.0  0.8 0.00 0.2000
[3,]  0.0  0.0 1.80 0.0500
[4,]  0.2  0.2 0.05 0.2125
```

For V2

```
U2<-W(V2)%*%Y
W2<-W(V2)%*%X
EY2<-solve(W(V2))%*%PX(W2)%*%U2
CB2<-C%*%ginv(t(W2)%*%W2)%*%t(W2)%*%U2
CovY2<-solve(W(V2))%*%PX(W2)%*%t(solve(W(V2)))
CovCB2<-C%*%ginv(t(W2)%*%W2)%*%t(C)
EY2
3.0    3.0    5.428571    5.428571    3.0    4.8    4.8
CB2
3.0    5.428571    -1.8    4.057143
CovY2
     [,1] [,2]    [,3]    [,4] [,5] [,6] [,7]
[1,]    1    1 0.00000 0.00000    0  0.0  0.0
[2,]    1    1 0.00000 0.00000    0  0.0  0.0
[3,]    0    0 0.42857 0.42857    0  0.0  0.0
[4,]    0    0 0.42857 0.42857    0  0.0  0.0
[5,]    0    0 0.00000 0.00000    1  0.0  0.0
[6,]    0    0 0.00000 0.00000    0  0.8  0.8
[7,]    0    0 0.00000 0.00000    0  0.8  0.8
```

```
CovCB2
     [,1]    [,2] [,3]    [,4]
[1,] 1.00 0.00000 0.00 0.25000
[2,] 0.00 0.42857 0.00 0.10714
[3,] 0.00 0.00000 1.80 0.05000
[4,] 0.25 0.10714 0.05 0.20179
```

PROBLEM 4:

a)

```
treat<-c(1,1,2,2,3,4,4)
V1inv<-1/ c(1,4,4,1,1,1,4)
BLUE1<-lm(formula=Y~factor(treat)-1,weights=V1inv)
BLUE1
   Call:
   lm(formula = Y ~ factor(treat) - 1, weights = V1inv)
   Coefficients:
   factor(treat)1  factor(treat)2  factor(treat)3  factor(treat)4
             2.6             5.6             3.0             4.8
```

b)

```
BLUE2<-lm.gls(formula = Y ~ factor(treat) - 1,W=V2, inverse = TRUE)
BLUE2
   $coefficients
   factor(treat)1 factor(treat)2 factor(treat)3 factor(treat)4
         3.000000       5.428571       3.000000       4.800000
   $residuals
   [1]  1.043382e-15 -2.000000e+00 -1.428571e+00  5.714286e-01  0.000000e+00  2.000000e-01 -
   8.000000e-01
   $effects
   factor(treat)1 factor(treat)2 factor(treat)3 factor(treat)4
      -3.0000000     -8.2922798     -3.0000000     -5.3665631      0.4472136      0.7559289      1.1547005
   $rank
   [1] 4
   $fitted.values
   [1] 3.000000 3.000000 5.428571 5.428571 3.000000 4.800000 4.800000
```

PROBLEM 5:

Recall from HW1 problem 4 that

SSE<- t(Y-Yhat)%*%(Y-Yhat) =4.5   and,

n<-7

r<-4   *rank(X)*

MSE<-SSE/(n-r)

a) A $(1-\alpha)\%$ confidence limits for σ^2 is given by $\left(\frac{SSE}{upper\,\frac{\alpha}{2}point\ of\ \chi^2_{n-rank(X)}}, \frac{SSE}{lower\,\frac{\alpha}{2}point\ of\ \chi^2_{n-rank(X)}}\right)$

```
a<-qchisq(0.05, df=n-r, ncp=0, lower.tail = TRUE, log.p = FALSE)
b<-qchisq(0.95, df=n-r, ncp=0, lower.tail = TRUE, log.p = FALSE)
upper<-SSE/a
lower<-SSE/b
```

A $(1-\alpha)\%$ confidence limits for σ is given by taking square roots to the limits of σ^2

```
sqrt(c(lower,upper))
```

(0.7588384,3.5762655)

A $(1-\alpha)\%$ confidence limits for $\widehat{c'\beta}$ is given by $\widehat{c'\beta} \pm t\sqrt{MSE}\sqrt{c'(X'X)^{-}c}$ where *t* is the upper $\frac{\alpha}{2}$ point of a $t_{n-rank(X)}$ distribution

```
tt<- qt(.95, df=n-r)
```

b)

```
C<-matrix(c(1,1/4,1/4,1/4,1/4),5,1,byrow=TRUE)
CB<-t(C)%*%ginv(t(X)%*%X)%*%t(X)%*%Y
lower<-CB-tt*sqrt(MSE)*sqrt(t(C)%*%ginv(t(X)%*%X)%*%C)
upper<-CB+tt*sqrt(MSE)*sqrt(t(C)%*%ginv(t(X)%*%X)%*%C)
```

(2.485683, 4.764317)

c)

```
C<-matrix(c(0,1,-1,0,0),5,1,byrow=TRUE)
CB<-t(C)%*%ginv(t(X)%*%X)%*%t(X)%*%Y
lower<-CB-tt*sqrt(MSE)*sqrt(t(C)%*%ginv(t(X)%*%X)%*%C)
upper<-CB+tt*sqrt(MSE)*sqrt(t(C)%*%ginv(t(X)%*%X)%*%C)
```

(-5.8822698, -0.1177302)

d)

```
C<- matrix(c(0,1,1,-1,-1),5,1,byrow=TRUE)
d<-0
l<-dim(C)[[2]]
CB<- t(C)%*%ginv(t(X)%*%X)%*%t(X)%*%Y
SSH0<-t(CB-d)%*%ginv(t(C)%*%ginv(t(X)%*%X)%*%C)%*%(CB-d)
F<-SSH0/l/MSE
F
0.06666667
pf(F, l, n-r, lower.tail =FALSE, log.p = FALSE)
```

0.8129535

A $(1-\alpha)\%$ prediciton limits for y^* is given by $\widehat{c'\beta} \pm t\sqrt{MSE}\sqrt{\gamma + c'(X'X)^{-}c}$ where *t* is the upper $\frac{\alpha}{2}$ point of a $t_{n-rank(X)}$ distribution

e)

```
C<-matrix(c(1,1,0,0,0),5,1,byrow=TRUE)
CB<-t(C)%*%ginv(t(X)%*%X)%*%t(X)%*%Y
g<-1/4  = γ
lower<-CB-tt*sqrt(MSE)*sqrt(g+t(C)%*%ginv(t(X)%*%X)%*%C)
upper<-CB+tt*sqrt(MSE)*sqrt(g+t(C)%*%ginv(t(X)%*%X)%*%C)
```

(-0.4961189, 4.4961189)

f)

```
C<-matrix(c(0,1,-1,0,0),5,1,byrow=TRUE)
CB<-t(C)%*%ginv(t(X)%*%X)%*%t(X)%*%Y
g<-2 = γ
lower<-CB-tt*sqrt(MSE)*sqrt(g+t(C)%*%ginv(t(X)%*%X)%*%C)
upper<-CB+tt*sqrt(MSE)*sqrt(g+t(C)%*%ginv(t(X)%*%X)%*%C)
```

c(-7.992238, 1.992238)

g)

```
C<- matrix(c(0,1,-1,0,0,0,1,0,-1,0,0,1,0,0,-1),3,5,byrow=TRUE)
d<-matrix(c(0,0,0),3,1)
l<-dim(C)[[1]]
CB<- C%*%ginv(t(X)%*%X)%*%t(X)%*%Y
SSH0<-t(CB-d)%*%ginv(C%*%ginv(t(X)%*%X)%*%t(C))%*%(CB-d)
F<-SSH0/l/MSE
F
2.428571
 pf(F, l, n-r, lower.tail = FALSE, log.p = FALSE)
```

0.2426305

h)

```
C<- matrix(c(0,1,-1,0,0,0,1,1,-1,-1),2,5,byrow=TRUE)
d<-matrix(c(4,0),2,1)
l<-dim(C)[[1]]
CB<- C%*%ginv(t(X)%*%X)%*%t(X)%*%Y
SSH0<-t(CB-d)%*%ginv(C%*%ginv(t(X)%*%X)%*%t(C))%*%(CB-d)
F<-SSH0/l/MSE
F
16.36667
 pf(F, l, n-r, lower.tail = TRUE, log.p = FALSE)
```

0.02432605