1. In class Vardeman claimed that hypotheses of the form $H_0 : \mathbf{C}\beta = \mathbf{0}$ can be written as $H_0 : E\mathbf{Y} \in C(\mathbf{X}_0)$ for $\mathbf{X}_0$ a suitable matrix (and $C(\mathbf{X}_0) \subset C(\mathbf{X})$). Let’s investigate this notion in the context of Problem 10 of Homework 1. Consider

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & .5 & .5 & -.5 & -.5 \\ 0 & 0 & 0 & 1 & -1 & .5 & -.5 & .5 & -.5 \end{pmatrix}$$

and the hypothesis $H_0 : \mathbf{C}\beta = \mathbf{0}$.

a) Find a matrix $\mathbf{A}$ such that $\mathbf{C} = \mathbf{AX}$.

b) Let

$$\mathbf{X}_0' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

Argue that the hypothesis under consideration is equivalent to the hypothesis $H_0 : E\mathbf{Y} \in C(\mathbf{X}_0)$. (Note: One clearly has $C(\mathbf{X}_0) \subset C(\mathbf{X})$. To show that $C(\mathbf{X}_0) \subset C(\mathbf{A}')^\perp$ it suffices to show that $\mathbf{P}_A\mathbf{X}_0 = \mathbf{0}$ and you can use $\mathbb{R}$ to do this. Then the dimension of $C(\mathbf{X}_0)$ is clearly 2, i.e. rank($\mathbf{X}_0$) = 2. So $C(\mathbf{X}_0)$ is a subspace of $C(\mathbf{X}) \cap C(\mathbf{A}')^\perp$ of dimension 2. But the dimension of $C(\mathbf{X}) \cap C(\mathbf{A}')^\perp$ is itself rank($\mathbf{X}$) − rank($\mathbf{C}$) = 4 − 2 = 2.)

c) Interpret the null hypothesis under discussion here in Stat 500 language.

2. Suppose we are operating under the (common Gauss-Markov) assumptions that $E\mathbf{e} = \mathbf{0}$ and $\text{Var}\mathbf{e} = \sigma^2 \mathbf{I}$.

a) Use fact 1. of Appendix 7.1 of the 2008 class outline to find $E(\mathbf{Y} - \hat{\mathbf{Y}})$ and $\text{Var}(\mathbf{Y} - \hat{\mathbf{Y}})$. (Use the fact that $\mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{P}_X)\mathbf{Y}$.) Then write

$$\begin{pmatrix} \hat{\mathbf{Y}} \\ \mathbf{Y} - \hat{\mathbf{Y}} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_X \\ \mathbf{I} - \mathbf{P}_X \end{pmatrix} \mathbf{Y}$$

and use fact 1 of Appendix 7.1 to argue that every entry of $\mathbf{Y} - \hat{\mathbf{Y}}$ is uncorrelated with every entry of $\hat{\mathbf{Y}}$.

b) Theorem 5.2.a of Rencher and Schaalje or Theorem 1.3.2 of Christensen say that if $E\mathbf{Y} = \mu$ and $\text{Var}\mathbf{Y} = \Sigma$ and $\mathbf{A}$ is a symmetric matrix of constants, then

$$E\mathbf{Y}'\mathbf{A}\mathbf{Y} = \text{tr}(\mathbf{A}\Sigma) + \mu'\mathbf{A}\mu$$

Use this fact and argue carefully that
\[
E\left( \mathbf{Y} - \hat{\mathbf{Y}} \right)^\prime \left( \mathbf{Y} - \hat{\mathbf{Y}} \right) = \sigma^2 \left( n - \text{rank} \left( \mathbf{X} \right) \right)
\]

3. a) In the context of Problem 3 of HW 1 and the fake data vector used in Problem 4 of HW 1, use \( \mathbb{R} \) and generalized least squares to find appropriate estimates for
\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 \\
1 & .25 & .25 & .25
\end{bmatrix}
\]
in the Aitken models with
\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}
\]
first \( \mathbf{V}_1 = \text{diag}(1,4,4,1,1,1,4) \) and then \( \mathbf{V}_2 = \]

(Do the necessary matrix calculations in \( \mathbb{R} \).)

b) For both of the above covariance structures, compare the (Aitken model) covariance matrices for generalized least squares estimators to (Aitken model) covariance matrices for the OLS estimators of \( \mathbf{E} \mathbf{Y} \) and the \( \mathbf{C} \beta \) above.

4. a) The basic \( \texttt{lm} \) function in \( \mathbb{R} \) allows one to automatically do weighted least squares, i.e. minimize \( \sum w_i (y_i - \hat{y}_i)^2 \) for positive weights \( w_i \). For the \( \mathbf{V}_1 \) case of the Aitken model of Problem 3, find the BLUEs of the 4 cell means using \( \texttt{lm} \) and an appropriate vector of weights. (Type > \texttt{help(lm)} in \( \mathbb{R} \) in order to get help with the syntax.)

b) The \( \texttt{lm.gls()} \) function in the \( \mathbb{R} \) contributed package \texttt{MASS} allows one to do generalized least squares as described in class. For the \( \mathbf{V}_2 \) case of the Aitken model of Problem 3, find the BLUEs of the 4 cell means using \( \texttt{lm.gls} \). (After loading the \texttt{MASS} package, Type > \texttt{help(lm.gls)} in \( \mathbb{R} \) in order to get help with the syntax.)

5. In the context of Problems 3 and 4 of HW #1, use \( \mathbb{R} \) matrix calculations to do the following in the (non-full-rank) Gauss-Markov normal linear model.
   a) Find 90% two-sided confidence limits for \( \sigma \).
   b) Find 90% two-sided confidence limits for \( \mu + \frac{1}{4} (\tau_1 + \tau_2 + \tau_3 + \tau_4) \).
   c) Find 90% two-sided confidence limits for \( \tau_1 - \tau_2 \).
d) Find a $p$-value for testing the null hypothesis

$$H_0 : \tau_1 + \tau_2 - \tau_3 - \tau_4 = 0 \text{ vs } H_a : \tau_1 + \tau_2 - \tau_3 - \tau_4 \neq 0.$$ 

e) Find 90% two-sided prediction limits for the sample mean of $n = 4$ future observations from the first set of conditions.

f) Find 90% two-sided prediction limits for the difference between a pair of future values, one from the first set of conditions (i.e. with mean $\mu + \tau_1$) and one from the second set of conditions (i.e. with mean $\mu + \tau_2$).

g) Find a $p$-value for testing

$$H_0 : \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

What is this test in the terminology of your previous statistical methods courses?

h) Find a $p$-value for testing

$$H_0 : \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$