Stat 511 HW#2 Spring 2009

1. In class Vardeman claimed that hypotheses of the form $\mathrm{H}_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ can be written as $\mathrm{H}_0 : \mathrm{E}\,\mathbf{Y} \in C(\mathbf{X}_0)$ for $\mathbf{X}_0$ a suitable matrix (and $C(\mathbf{X}_0) \subset C(\mathbf{X})$). Let's investigate this notion in the context of Problem 10 of Homework 1. Consider

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & .5 & .5 & -.5 & -.5 \\ 0 & 0 & 0 & 1 & -1 & .5 & -.5 & .5 & -.5 \end{pmatrix}$$

and the hypothesis $\mathrm{H}_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$.

a) Find a matrix $\mathbf{A}$ such that $\mathbf{C} = \mathbf{A}\mathbf{X}$.

b) Let

$$\mathbf{X}_0' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Argue that the hypothesis under consideration is equivalent to the hypothesis $\mathrm{H}_0 : \mathrm{E}\,\mathbf{Y} \in C(\mathbf{X}_0)$. (Note: One clearly has $C(\mathbf{X}_0) \subset C(\mathbf{X})$. To show that $C(\mathbf{X}_0) \subset C(\mathbf{A}')^{\perp}$ it suffices to show that $\mathbf{P}_{\mathbf{A}'}\mathbf{X}_0 = \mathbf{0}$ and you can use R to do this. Then the dimension of $C(\mathbf{X}_0)$ is clearly 2, i.e. $\mathrm{rank}(\mathbf{X}_0) = 2$. So $C(\mathbf{X}_0)$ is a subspace of $C(\mathbf{X}) \cap C(\mathbf{A}')^{\perp}$ of dimension 2. But the dimension of $C(\mathbf{X}) \cap C(\mathbf{A}')^{\perp}$ is itself $\mathrm{rank}(\mathbf{X}) - \mathrm{rank}(\mathbf{C}) = 4 - 2 = 2$.)

c) Interpret the null hypothesis under discussion here in Stat 500 language.

2. Suppose we are operating under the (common Gauss-Markov) assumptions that $\mathrm{E}\,\boldsymbol{\varepsilon} = \mathbf{0}$ and $\mathrm{Var}\,\boldsymbol{\varepsilon} = \sigma^2\mathbf{I}$.

a) Use fact 1. of Appendix 7.1 of the 2008 class outline to find $\mathrm{E}(\mathbf{Y} - \hat{\mathbf{Y}})$ and $\mathrm{Var}(\mathbf{Y} - \hat{\mathbf{Y}})$. (Use the fact that $\mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{P}_{\mathbf{X}})\mathbf{Y}$.) Then write

$$\begin{pmatrix} \hat{\mathbf{Y}} \\ \mathbf{Y} - \hat{\mathbf{Y}} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{\mathbf{X}} \\ \mathbf{I} - \mathbf{P}_{\mathbf{X}} \end{pmatrix} \mathbf{Y}$$

and use fact 1 of Appendix 7.1 to argue that every entry of $\mathbf{Y} - \hat{\mathbf{Y}}$ is uncorrelated with every entry of $\hat{\mathbf{Y}}$.

b) Theorem 5.2.a of Rencher and Schaalje or Theorem 1.3.2 of Christensen say that if $\mathrm{E}\mathbf{Y} = \boldsymbol{\mu}$ and $\mathrm{Var}\,\mathbf{Y} = \boldsymbol{\Sigma}$ and $\mathbf{A}$ is a symmetric matrix of constants, then

$$\mathrm{E}\,\mathbf{Y}'\mathbf{A}\mathbf{Y} = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$$

Use this fact and argue carefully that

$$\mathrm{E}\left(\mathbf{Y}-\hat{\mathbf{Y}}\right)'\left(\mathbf{Y}-\hat{\mathbf{Y}}\right)=\sigma^2\left(n-\mathrm{rank}\left(\mathbf{X}\right)\right)$$

3. a) In the context of Problem 3 of HW 1 and the fake data vector used in Problem 4 of HW 1, use R and generalized least squares to find appropriate estimates for

$$\mathrm{E}\,\mathbf{Y} \text{ and } \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & .25 & .25 & .25 & .25 \end{bmatrix}\boldsymbol{\beta}$$

in the Aitken models with

$$\text{first } \mathbf{V}_1=\mathrm{diag}\left(1,4,4,1,1,1,4\right) \text{ and then } \mathbf{V}_2=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(Do the necessary matrix calculations in R.)

b) For both of the above covariance structures, compare the (Aitken model) covariance matrices for generalized least squares estimators to (Aitken model) covariance matrices for the OLS estimators of $\mathrm{E}\,\mathbf{Y}$ and the $\mathbf{C}\boldsymbol{\beta}$ above.

4. a) The basic `lm` function in R allows one to automatically do weighted least squares, i.e. minimize $\sum w_i\left(y_i-\hat{y}_i\right)^2$ for positive weights w_i. For the $\mathbf{V}_1$ case of the Aitken model of Problem 3, find the BLUEs of the 4 cell means using `lm` and an appropriate vector of weights. (Type `> help(lm)` in R in order to get help with the syntax.)

b) The `lm.gls()` function in the R contributed package MASS allows one to do generalized least squares as described in class. For the $\mathbf{V}_2$ case of the Aitken model of Problem 3, find the BLUEs of the 4 cell means using `lm.gls`. (After loading the MASS package, Type `> help(lm.gls)` in order to get help with the syntax.)

5. In the context of Problems 3 and 4 of HW #1, use R matrix calculations to do the following in the (non-full-rank) Gauss-Markov normal linear model.

a) Find 90% two-sided confidence limits for σ.

b) Find 90% two-sided confidence limits for $\mu+\frac{1}{4}\left(\tau_1+\tau_2+\tau_3+\tau_4\right)$.

c) Find 90% two-sided confidence limits for $\tau_1-\tau_2$.

d) Find a *p*-value for testing the null hypothesis $\mathrm{H}_0: \tau_1 + \tau_2 - \tau_3 - \tau_4 = 0$ vs $\mathrm{H}_\mathrm{a}: \tau_1 + \tau_2 - \tau_3 - \tau_4 \neq 0$.

e) Find 90% two-sided prediction limits for the sample mean of $n = 4$ future observations from the first set of conditions.

f) Find 90% two-sided prediction limits for the difference between a pair of future values, one from the first set of conditions (i.e. with mean $\mu + \tau_1$) and one from the second set of conditions (i.e. with mean $\mu + \tau_2$).

g) Find a *p*-value for testing $\mathrm{H}_0\!: \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. What is this test in the terminology of your previous statistical methods courses?

h) Find a *p*-value for testing $\mathrm{H}_0\!: \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.