I have neither given nor received unauthorized assistance on this exam.

KEY (corrected several times!)

Name

Name Printed
1. An experimental data set in a set of slides due to S.A. Jenekhe found on the University of Washington Chemistry Department web site of Prof. Lawrence Ricker concerns CO₂ solubility in a glassy polymer. Given are the pressures, $p$, and corresponding concentrations, $c$, of CO₂ below.

<table>
<thead>
<tr>
<th>Pressure, $p$ (atm)</th>
<th>2.74</th>
<th>6.10</th>
<th>9.76</th>
<th>14.45</th>
<th>18.92</th>
<th>26.74</th>
<th>33.28</th>
<th>42.23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concentration, $c$ (cm³ (STP)/cm³ polymer)</td>
<td>36.6</td>
<td>51.4</td>
<td>64.3</td>
<td>78.7</td>
<td>91.5</td>
<td>110.3</td>
<td>122.9</td>
<td>143.7</td>
</tr>
</tbody>
</table>

A standard deterministic model for gas solubility in a polymer is

$$
c = Hp + L_c \left( \frac{L_a p}{1 + L_a p} \right)$$

for constants $H$ (the Henry's law constant), $L_c$ (the Langmuir capacity constant), and $L_a$ (the Langmuir affinity constant). Below is a plot of these data and a fitted concentration versus pressure curve. Note that for large pressure this (fitted) curve is nearly linear with slope $H$ and intercept $L_c$, while the derivative of concentration with respect to pressure at 0 pressure is $H + L_a \cdot I_a$.

There is an R printout at the end of this exam from the session in which this plot was generated. Use it to answer the following questions about a nonlinear regression analysis of this situation based on a model

$$
c_i = H p_i + L_c \left( \frac{L_a p_i}{1 + L_a p_i} \right) + \varepsilon_i$$

for iid $N(0, \sigma^2)$ errors $\varepsilon_i$, $i = 1, 2, \ldots, 8$. 

(*)
a) What are approximate 95\% confidence limits for the standard deviation of concentration at any fixed pressure according to the model (*)? (If you need some percentage point(s) of a distribution that you don't have, say very carefully/completely exactly what you need.) Plug into any formula you provide.

Carrying over the LM method to this non-linear model context, use

\[ s = \sqrt{\frac{\text{SSE}}{n-k}} = 1.438 \] with "d.f. = n-k = 5

and limits

\[ \left( s \sqrt{\frac{n-k}{X^2_{\text{upper}}}}, s \sqrt{\frac{n-k}{X^2_{\text{lower}}}} \right) \]

i.e.

\[ \left( 1.438 \sqrt{\frac{5}{12.833}}, 1.438 \sqrt{\frac{5}{83.1}} \right) \]

b) Under what conditions on the parameters of model (*) is the mean concentration a simple multiple of pressure? Is there definitive evidence in these data that such a ("single mode") model is too simple and so the full complexity of model (*) is justified? Explain in terms of some measures of statistical significance.

\[ \hat{c} = \text{constant} \times \hat{p} \] exactly when \( L_c = 0 \) or \( L_a = 0 \). But it's clear from the printout that both \( H_0 : L_c = 0 \) and \( H_0 : L_a = 0 \) produce very small p-values (for t-tests of these hypotheses).

It's thus clear that a "single mode" model is not adequate for describing these data and the full complexity of model (*) is justified.
c) What are approximate 95% confidence limits for the derivative of mean concentration with respect to pressure at 0 pressure? (Plug into an appropriate formula. You don't need to do arithmetic, but you must plug in, and if you don't have necessary percentage points of a distribution, say very carefully/completely exactly what you need.)

This is $H + \hat{L}_c L_a$. Note that \( \frac{\partial}{\partial H} (H + L_c L_a) = 1 \), \( \frac{\partial}{\partial L_c} (H + L_c L_a) = L_a \) and \( \frac{\partial}{\partial L_a} (H + L_c L_a) = L_c \). Then

\[
\text{Var} \left( \hat{H} + \hat{L}_c \hat{L}_a \right) \approx \begin{pmatrix} L_a & L_c \end{pmatrix} \text{Var} \left( \begin{pmatrix} \hat{L}_a \\ \hat{L}_c \end{pmatrix} \right) \begin{pmatrix} 1 \\ L_a \\ L_c \end{pmatrix}
\]

and a plug-in estimate of the thing on the right is

\[
\begin{pmatrix} 1, \hat{L}_a, \hat{L}_c \end{pmatrix} \text{Var} \left( \begin{pmatrix} \hat{L}_c \\ \hat{L}_a \end{pmatrix} \right) \begin{pmatrix} 1 \\ \hat{L}_a \\ \hat{L}_c \end{pmatrix}
\]

so approximate confidence limits are

\[
\left[ 2.179 + (55.043)(0.4183) \right]
\]

\[
\pm 2.571 \sqrt{\begin{pmatrix} 1 \\ \frac{4183}{55.043} \end{pmatrix} \begin{pmatrix} 0.0074 & -0.2862 & 0.0057 \\ -0.2862 & 12.2056 & -0.2595 \\ 0.0057 & -0.2595 & 0.0065 \end{pmatrix} \begin{pmatrix} 1 \\ 4183 \\ 55.043 \end{pmatrix}}
\]

2. In a typical industrial "gauge R&R study," each of \( I \) different parts from some process is measured \( m \) times by each of \( J \) different operators, as a way of studying the consistency of measurement using a single gauge. We will here consider a case where the operators are "fixed" (being the only ones a company will ever use to do such measuring) while parts are "random" (representing ongoing production of such parts) and for

\[ y_{ijk} = \text{the } k\text{th measurement obtained on part } i \text{ by operator } j \]

model as

\[
y_{ijk} = \mu + \alpha_i + \beta_j + \alpha \beta_j + \epsilon_{ijk} \tag{**}
\]

where \( \mu \) and the \( \beta_j \) are unknown constants and the \( \alpha_i, \alpha \beta_j, \) and \( \epsilon_{ijk} \) are independent random variables, with \( \alpha_i \sim \text{iid } N(0, \sigma_{\alpha}^2) \), \( \alpha \beta_j \sim \text{iid } N(0, \sigma_{\alpha \beta}^2) \), and \( \epsilon_{ijk} \sim \text{iid } N(0, \sigma^2) \). (Here the \( \beta_j \) might be thought of as consistent operator biases and the \( \alpha \beta_j \) might be thought of as so-called operator "nonlinearities of measurement.")

To begin, first consider a small/toy case where \( I = J = m = 2 \) (there are 2 parts, 2 operators, and each part is measured 2 times by each operator).
a) For 8 observations written down in dictionary order, show how to write out model (**') in mixed linear model matrix form (by providing the elements of $Y = X\beta + Zu + \varepsilon$ indicated below).

$$Y = \begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_{11} \\ \beta_{12} \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{11} \\ \varepsilon_{12} \end{pmatrix}$$

b) Write out the following in terms of model (**') parameters.

$$\text{Var} y_{111} = \frac{\sigma^2}{\alpha} + \frac{\sigma^2}{\alpha \beta} + \sigma^2$$

$$\text{Cov}(y_{111}, y_{112}) = \frac{\sigma^2}{\alpha} + \frac{\sigma^2}{\alpha \beta}$$

$$\text{Cov}(y_{111}, y_{121}) = \frac{\sigma^2}{\alpha}$$

$$\text{Cov}(y_{111}, y_{211}) =$$

At the end of this exam, there is an R printout for an analysis based on model (**') of a modification of a real data set from an R&R study based on $I = 4$ parts, $J = 3$ operators, and $m = 2$ measurements per part. (These are measured heights of some steel punches in $10^{-3}$ inch.) Use it to answer the next two questions.
c) Based on the results on the printout

- do you find clear evidence of differences in operator measurement biases, and
- do operator "nonlinearities" appear to play a large role in measurement of these punch heights?

(Return to the parenthetical remark following model statement (**)) for use of these terms.) Explain using appropriate values from the printout.

1st: p-values for \( H_0: \beta_1 = \beta_2 \) and \( H_0: \beta_1 = \beta_3 \) are big. Thus there is no clear evidence of differences in biases in these pairs. Further, the apparent difference between \( \beta_2 \) and \( \beta_3 \) is even smaller in magnitude than that between either of these and \( \beta_1 \) (\( \beta_2 - \beta_1 \) and \( \beta_3 - \beta_1 \) have the same sign). So p-value for \( H_0: \beta_2 = \beta_3 \) will also be big. No clear evidence.

2nd: \( \sigma_\beta \) appears to be small in comparison to \( \sigma \) and \( \sigma_\epsilon \). It seems that "non-linearities" play a relatively small role in this measurement scenario.

d) What are approximate BLUPs for

- \( \mu + \alpha_i + \beta_1 + \alpha \beta_{i1} \) (a long-run average of measurements of part 1 by operator 1) (give a numerical value)

\[
\frac{500.5 \text{BLUP}}{500.5367} = \frac{498.625 + 2.2247 + (-.31305)}{\alpha + \beta_{i1}}
\]

- \( \mu + \alpha_i + \beta_1 + \alpha \beta_{i1} + \epsilon_{si1} \) (a measurement on a new punch by operator #1) Here, give both the BLUP AND an appropriate standard error (give numerical values).

\[
\hat{\mu} + \hat{\beta}_1 + \hat{\alpha}_s + \hat{\alpha} \beta_{s1} + \hat{\epsilon}_{s11} = 498.625 + 0 + 0 + 0 = 498.625
\]

\[
\text{Var}((\hat{\mu} + \hat{\alpha}_s + \hat{\beta}_1 + \hat{\alpha} \beta_{s1} + \hat{\epsilon}_{s11}) - (\hat{\mu} + \hat{\beta}_1)) = \text{Var}(\hat{\mu} + \hat{\beta}_1) + \sigma^2 + \sigma^2_\alpha \beta + \sigma^2
\]

So a prediction standard error is

\[
\sqrt{\text{Var}((\hat{\mu} + \hat{\beta}_1 + \hat{\alpha}_s + \hat{\alpha} \beta_{s1} + \hat{\epsilon}_{s11}) - (\hat{\mu} + \hat{\beta}_1))} = \sqrt{(3.021) + (4.930)^2 + (1.5964)^2 + (0.9129)^2}
\]
3. Suppose that for \( i = 1, 2 \) and \( j = 1, 2 \), \( y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \) for independent variables

\[
\alpha_i \sim \text{iid} \, N\left(0, \sigma^2_{\alpha}\right) \quad \text{and} \quad \varepsilon_{ij} \sim \text{iid} \, N\left(0, \sigma^2\right).
\]

Take \( \mathbf{W} = \mathbf{BY} \) for \( \mathbf{Y} = (y_{11}, y_{12}, y_{21}, y_{22})' \) and

\[
\mathbf{B} = \begin{pmatrix}
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}.
\]

a) Argue that REML estimation of \( \sigma^2_{\alpha} \) and \( \sigma^2 \) can be based on \( \mathbf{W} \) and write out explicitly the function of \( w_1, w_2, w_3, \sigma^2_{\alpha}, \) and \( \sigma^2 \) that can be maximized as a function of \( \sigma^2_{\alpha} \) and \( \sigma^2 \) to produce REML estimates. Since each row sum in \( \mathbf{B} \) is 0, it's clear that \( \mathbf{B}' \mathbf{1} = \mathbf{B} \mathbf{X} = 0 \). Further rows of \( \mathbf{B} \) are easily seen to be mutually perpendicular, so none is a l.e. of the other 2, i.e. \( \mathbf{B} \) is of rank 3 and thus full rank. So we can do REML estimation based on \( \mathbf{W} \). Now \( \text{Var} \, \mathbf{Y} = \sigma^2_{\alpha} \mathbf{I} \otimes \mathbf{J} + \sigma^2 \mathbf{I} \) and \( \text{Var} \, \mathbf{W} = \mathbf{B} \, \text{Var} \, \mathbf{Y} \, \mathbf{B}' \). Thus \( \text{Var} \, \mathbf{W} = \text{diag} \left( 8\sigma^2_{\alpha} + 4\sigma^2, 2\sigma^2, 2\sigma^2 \right) \) and the restricted likelihood is then

\[
L_r(\sigma^2) = \left(2\pi\right)^{-3/2} \left(8\sigma^2_{\alpha} + 4\sigma^2 \right)^{1/2} \sigma^2 \cdot 2 \sigma^2 \left[ w_1^2 / \left(8\sigma^2_{\alpha} + 4\sigma^2 \right) + (w_2^2 + w_3^2) / 2\sigma^2 \right]^{-1/2} \cdot \exp \left(-\frac{1}{2} \left[ w_1^2 / \left(8\sigma^2_{\alpha} + 4\sigma^2 \right) + (w_2^2 + w_3^2) / 2\sigma^2 \right] \right).
\]

b) The restricted loglikelihood from a) can be written as a function of log-variances \( \gamma_1 = \log \sigma^2_{\alpha} \) and \( \gamma_2 = \log \sigma^2 \). For a particular \( \mathbf{W} \), this is maximized at \( \hat{\gamma}_1 = .6931 \) and \( \hat{\gamma}_2 = 1.3863 \). The Hessian (matrix of second partial derivatives) of this function at its maximizer is

\[
\begin{pmatrix}
-1.5 & -1 \\
-1 & -2
\end{pmatrix}.
\]

Find approximate 95% confidence limits for \( \sigma^2_{\alpha} \) based on this information. (Plug in and evaluate.) The inverse of the Hessian functions as an estimated covariance matrix for \((\hat{\gamma}_1, \hat{\gamma}_2)'\). This is

\[
\begin{pmatrix}
1 & -.5 \\
-.5 & .75
\end{pmatrix}
\]

so approximate 95% limits for \( \gamma_2 \) are

\[
1.39 \pm 1.96 \sqrt{.75}
\]

and thus 95% limits for \( \sigma^2_{\alpha} \) are \((\exp(1.39-1.96\sqrt{.75}), \exp(1.39+1.96\sqrt{.75}))\) and limits for \( \sigma^2 \) are

\[
(\exp(\frac{1}{2}(1.39-1.96\sqrt{.75})), \exp(\frac{1}{2}(1.39+1.96\sqrt{.75})))
\]
4. Suppose that for \( i = 1,2,3 \) independent observations \( y_{ij} \sim \text{iid } N(\mu_i, \sigma^2_i) \) for \( j = 1,\ldots,n_i \) (that is, we have independent samples of sizes \( n_i \) from three different normal distributions). For constants \( c_1, c_2, \) and \( c_3 \), the random variable \( c_1 \bar{Y}_1 + c_2 \bar{Y}_2 + c_3 \bar{Y}_3 \) has variance

\[
V = \frac{c_1^2}{n_1} \sigma_1^2 + \frac{c_2^2}{n_2} \sigma_2^2 + \frac{c_3^2}{n_3} \sigma_3^2
\]

Based on variances for the 3 samples \((s_1^2, s_2^2, \text{ and } s_3^2)\) use the Cochran-Satterthwaite approximation to identify approximate 95% confidence limits for \( V \). (Say explicitly what percentage point(s) of exactly what distribution will be needed.)

The sample variances are independent mean squares and the Cochran-Satterthwaite estimate of \( V \) is

\[
s^2 = \sum_{i=1}^{3} \frac{c_i^2}{n_i} s_i^2
\]

The "approximate d.f." for this is

\[
\hat{\nu} = \frac{(s^2)^2}{\sum_{i=1}^{3} \left( \frac{c_i^2}{n_i} s_i^2 \right)^2}
\]

(you probably should round down to an integer)

and then for \( \chi^2_{\text{upper}} \) and \( \chi^2_{\text{lower}} \) being the upper and lower 2.5% pts of the \( \chi^2 \) dsn with d.f. \( \hat{\nu} \) use limits

\[
\left(\frac{s^2}{\chi^2_{\text{upper}}}, \frac{s^2}{\chi^2_{\text{lower}}}\right)
\]
For Problem #1

> pressure<-c(2.74, 6.10, 9.76, 14.45, 18.92, 26.74, 33.28, 42.23)
> conc<-c(36.6, 51.4, 64.3, 78.7, 91.5, 110.3, 122.9, 143.7)

> nlrfit<-nls(formula=conc~H*pressure+LC*((LC*pressure)/(1+LC*pressure)),start=c(H=2,LC=50,LA=.5),trace=T)
446.4045 : 2.0 50.0 0.5
11.02893 : 2.1865348 54.6735257 0.4116493
10.34221 : 2.1785177 55.0649938 0.4177164
10.34211 : 2.1789683 55.0447196 0.4182403
10.34211 : 2.1790082 55.0429003 0.4182864
10.34211 : 2.1790117 55.0427404 0.4182904

> summary(nlrfit)

Formula: conc ~ H * pressure + ((LC * LA)/(1 + LA * pressure)) * pressure

Parameters:

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| H        | 2.179001   | 0.08574 | 25.414   | 1.76e-06 *** |
| LC       | 55.04274   | 3.49365 | 15.755   | 1.87e-05 *** |
| LA       | 0.41829    | 0.08051 | 5.195    | 0.00348 **   |

--

Signif. codes:  0 ***  0.001 ***  0.01 **  0.05 *  0.1  1

Residual standard error: 1.438 on 5 degrees of freedom

Number of iterations to convergence: 5
Achieved convergence tolerance: 1.959e-06

> confint(nlrfit)

Waiting for profiling to be done...
51.67906 : 55.0427404 0.4182904
13.45704 : 58.8543577 0.3425121

:
:
:

47.62486 : 2.455613 43.3577777
64.20141 : 2.523317 40.680263
61.44185 : 2.497246 41.751108
79.19115 : 2.562831 39.220090
76.03984 : 2.53446 40.34596
94.31986 : 2.598318 37.934771
90.55775 : 2.566830 39.146689

2.5% 97.5%

H 1.9317119 2.3913757
LC 46.7938116 65.8979832
LA 0.2616633 0.7487824

> vcov(nlrfit)

       H     LA
H 0.0079
LA 0.0079 0.0035
For Problem #2

```r
> height
> Part
 [1]  1  1  1  1  1  2  2  2  2  2  2  3  3  3  3  3  3  3  3  4  4  4  4  4  4  4
Levels:  1  2  3  4
> Operator
 [1]  1  1  2  2  3  3  1  1  2  2  3  3  1  1  2  2  3  3
Levels:  1  2  3
> PartbyOperator
 [1]  1  1  2  2  3  3  1  1  2  2  3  3  1  1  2  2  3  3
Levels:  1  2  3

> lmeRandR<-lme(height~1+Operator,random=~1|Part/PartbyOperator)

> summary(lmeRandR)
Linear mixed-effects model fit by REML
Data: NULL

       AIC     BIC   logLik
85.73936 92.0065 -36.86968

Random effects:
Formula: ~1 | Part
 (Intercept) StdDev: 1.596437

Formula: ~1 | PartbyOperator %in% Part
 (Intercept) Residual
StdDev: 0.4930067 0.9128709

Fixed effects: height ~ 1 + Operator

Value Std.Error DF  t-value p-value
(Intercept) 498.625 0.8955912 12  556.75 0.0000
Operator2  -1.000  0.5743354  6  -1.7411  0.1323
Operator3  -0.625  0.5743354  6  -1.0882  0.3183

Correlation:
(Intr) Oprtr2
Operator2  -0.321
Operator3  -0.321  0.500

Standardized Within-Group Residuals:
          Min     Q1    Med     Q3     Max
-1.89128663 -0.54044333 0.57924966 0.57924966 1.89128663

Number of Observations: 24
Number of Groups:
 Part PartbyOperator %in% Part
        4           12
```

---

For $H_0: \beta_3 - \beta_1 = 0$, inference for $\beta_j$'s under the R sum/baseline restriction. For $H_0: \beta_2 - \beta_1 = 0$.
> fixed.effects(lmeRandR)
(Intercept)  Operator2  Operator3
  498.625      -1.000     -0.625

> vcov(lmeRandR)
(Intercept)  Operator2  Operator3
 (Intercept)   0.8020835 -0.1649306 -0.1649306
Operator2      -0.1649306   0.3298611   0.1649306
Operator3      -0.1649306   0.1649306   0.3298611

> intervals(lmeRandR)
Approximate 95% confidence intervals

Fixed effects:

<table>
<thead>
<tr>
<th></th>
<th>lower</th>
<th>est.</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>496.673675</td>
<td>498.625</td>
<td>500.576325</td>
</tr>
<tr>
<td>Operator2</td>
<td>-2.405348</td>
<td>-1.000</td>
<td>0.405348</td>
</tr>
<tr>
<td>Operator3</td>
<td>-2.030348</td>
<td>-0.625</td>
<td>0.780348</td>
</tr>
</tbody>
</table>

attr(,"label")
[1] "Fixed effects:

Random Effects:
Level: Part

<table>
<thead>
<tr>
<th></th>
<th>lower</th>
<th>est.</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>sd((Intercept))</td>
<td>0.6722698</td>
<td>1.596437</td>
<td>3.791055</td>
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</tbody>
</table>

Level: PartbyOperator

<table>
<thead>
<tr>
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<th>est.</th>
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</tr>
</thead>
<tbody>
<tr>
<td>sd((Intercept))</td>
<td>0.09020527</td>
<td>0.4930067</td>
<td>2.694472</td>
</tr>
</tbody>
</table>

Within-group standard error:

<table>
<thead>
<tr>
<th></th>
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<th>est.</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5976521</td>
<td>0.9128709</td>
<td>1.3943452</td>
</tr>
</tbody>
</table>

> random.effects(lmeRandR)
Level: Part

<table>
<thead>
<tr>
<th></th>
<th>lower</th>
<th>est.</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>2.2247074</td>
<td>-0.2301421</td>
<td>-1.1507107</td>
</tr>
<tr>
<td></td>
<td>-0.8438545</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Level: PartbyOperator %in% Part

<table>
<thead>
<tr>
<th></th>
<th>lower</th>
<th>est.</th>
<th>upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-0.313050149</td>
<td>-0.423792081</td>
<td>0.101423605</td>
</tr>
<tr>
<td></td>
<td>0.038736586</td>
<td>-0.145473971</td>
<td>0.084789226</td>
</tr>
<tr>
<td></td>
<td>0.193682932</td>
<td>0.009472374</td>
<td></td>
</tr>
</tbody>
</table>
\[
\begin{array}{cccc}
3/3 & -0.312896102 \\
4/1 & 0.080630631 \\
4/2 & -0.287790484 \\
4/3 & 0.126683270 \\
\end{array}
\]

```r
> predict(lmeRandR, level=0:2)
   Part PartbyOperator predict.fixed predict.Part predict.PartbyOperator
1   1     1/1         498.625        500.8497         500.5367
2   1     1/1         498.625        500.8497         500.5367
3   1     1/2         497.625        499.8497         500.2735
4   1     1/2         497.625        499.8497         500.2735
5   1     1/3         498.000        500.2247         500.3261
6   1     1/3         498.000        500.2247         500.3261
7   2     2/1         498.625        498.3949         498.4336
8   2     2/1         498.625        498.3949         498.4336
9   2     2/2         497.625        497.3949         497.2494
10  2     2/2         497.625        497.3949         497.2494
11  2     2/3         498.000        497.7699         497.8546
12  2     2/3         498.000        497.7699         497.8546
13  3     3/1         498.625        497.4743         497.6680
14  3     3/1         498.625        497.4743         497.6680
15  3     3/2         497.625        496.4743         496.4838
16  3     3/2         497.625        496.4743         496.4838
17  3     3/3         498.000        496.8493         496.5364
18  3     3/3         498.000        496.8493         496.5364
19  4     4/1         498.625        497.7811         497.8618
20  4     4/1         498.625        497.7811         497.8618
21  4     4/2         497.625        496.7811         496.4934
22  4     4/2         497.625        496.7811         496.4934
23  4     4/3         498.000        497.1561         497.2828
24  4     4/3         498.000        497.1561         497.2828
\end{array}
```